SHORT PROOF OF A THEOREM OF RADO ON GRAPHS

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Berge's proof [1, p. 18] of Rado's theorem, a special case of the lemma in [2, p. 337], suffers from inaccuracies. In this note the result is derived from the following lemma of König [1, p. 17]. If $(A_1, A_2, \cdots)$ is a sequence of nonempty, pairwise disjoint finite sets and $<$ is any relation between elements of consecutive sets such that for all $x_n \in A_n$, an element $x_{n-1} \in A_{n-1}$ exists with $x_{n-1} < x_n$, then a sequence $(a_1, a_2, \cdots)$ exists with $a_n \in A_n$ for all $n$, such that $a_1 < a_2 < \cdots < a_n < \cdots$.

Theorem (Rado). Given a locally finite graph, $^1 (G, \Gamma)$ a finite set of integers $K$ and a mapping $T$ of subsets of $K$ into subsets of $K$; if each finite subgraph, $(A, \Gamma_A)$, admits a function $\phi_a$ such that

$$\phi_a(x) \in T\{\phi_a(\Gamma_A x)\}, \quad \text{for all } x \in A,$$

then $(G, \Gamma)$ admits a function $\phi$ such that

$$\phi(x) \in T\{\phi(\Gamma x)\}, \quad \text{for all } x \in G.$$  

Proof. Since there is no interaction between connected components, we may take $G$ to be connected (hence countable, because of local finiteness [1, p. 18]). Let $G = \{x_1, x_2, \cdots\}$ and define a sequence of subsets of $G$ by setting $G_n = \{x_1, x_2, \cdots, x_{p_n}\}$, where $p_1$ is the least integer $r$ such that $r > 1$ and $\{x_1, x_2, \cdots, x_r\}$ contains $\Gamma x_1$, and $p_n$, for $n > 1$, is the least integer $r$ such that $r > p_{n-1}$ and $\{x_1, x_2, \cdots, x_r\}$ contains $\Gamma x_1 \cup \Gamma x_2 \cup \cdots \cup \Gamma x_n$. For each $n$, let $A_n$ be the set of all mappings $\phi: G_n \rightarrow K$ such that $\phi(x_i) \in T\{\phi(\Gamma x_i)\}$, for $i = 1, 2, \cdots, n$. Each $A_n$ is nonempty since it contains $\phi_{a_n}$, which exists by hypothesis. Moreover, each $A_n$ is finite and $A_n$ and $A_m$ are disjoint for $n \neq m$, since $G_n \neq G_m$. Define a relation between mappings in $A_{n-1}$ and $A_n$ by setting $\phi < \psi$ if and only if $\psi|_{A_{n-1}} = \phi$, where $\psi|_{A_{n-1}}$ is the restriction of $\psi$ to $G_{n-1}$. Then, for each $\psi \in A_n$, there is a $\phi \in A_{n-1}$ with $\phi < \psi$, in fact, $\phi = \psi|_{A_{n-1}}$ will do. Thus, by König's result, there is a sequence $\phi_1 < \phi_2 < \cdots < \phi_n < \cdots$, with $\phi_n \in A_n$, for all $n$. Now define $\phi$ on all of $G$ by $\phi(x_n) = \phi_n(x_n)$, for all $x_n \in G$. It is immediately verified that $\phi$ has the required property.

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A graph $(G, \Gamma)$, where $\Gamma$ maps elements of $G$ into subsets of $G$, is called locally finite if $\Gamma x$ and $\Gamma^{-1}x$ are finite, for every $x \in G$. 

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CONCERNING CONTINUOUS IMAGES OF COMPACT ORDERED SPACES

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It is the purpose of this paper to prove that if each of $X$ and $Y$ is a compact Hausdorff space containing infinitely many points, and $X \times Y$ is the continuous image of a compact ordered space $L$, then both $X$ and $Y$ are metrizable. The preceding theorem is a generalization of a theorem [1] by Mardešić and Papić, who assume that $X$, $Y$, and $L$ are also connected. Young, in [3], shows that the Cartesian product of a “long” interval and a real interval is not the continuous image of any compact ordered space.

In this paper, the word compact is used in the “finite cover” sense. The phrase “ordered space” means a totally ordered topological space with the order topology. A subset $M$ of a topological space is said to be hereditarily separable provided each subset of $M$ is separable. If $a$ and $b$ are points of an ordered space $L$ and $a<b$, then $[a, b]$, $(a, b)$ will denote the set of all points $x$ of $L$ such that $a \leq x \leq b$ ($a<x<b$), provided there is one; also, $[a, b]$ will be used even in the case where $a = b$. A subset $M$ of an ordered space $L$ is convex provided that if $a \in M$, $b \in M$, and $a < b$, then $[a, b] \subseteq M$. If $M$ is a subset of an ordered space $L$, then $G(M)$ will denote the set of all ordered pairs $(a, b)$ such that (1) $a \in M$, $b \in M$, and $a < b$, and (2) $\{a, b\} = M \cdot [a, b]$, provided there is one.

**Lemma 0.** If $M$ is a compact subset of the ordered space $L$, then the relative topology of $L$ on $M$ is the same as the order topology on $M.

References


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