SHORT PROOF OF A THEOREM OF RADO ON GRAPHS

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Berge's proof [1, p. 18] of Rado's theorem, a special case of the lemma in [2, p. 337], suffers from inaccuracies. In this note the result is derived from the following lemma of König [1, p. 17]. If $(A_1, A_2, \cdots)$ is a sequence of nonempty, pairwise disjoint finite sets and $<$ is any relation between elements of consecutive sets such that for all $x_n \in A_n$, an element $x_{n-1} \in A_{n-1}$ exists with $x_{n-1} < x_n$, then a sequence $(a_1, a_2, \cdots)$ exists with $a_n \in A_n$, for all n, such that $a_1 < a_2 < \cdots < a_n < \cdots$.

THEOREM (RADO). *Given a locally finite graph,*[1] (G, Γ) *a finite set of integers* K *and a mapping* T *of subsets of* K *into subsets of* K; *if each finite subgraph,* (A, Γ_A), *admits a function* ϕ_A *such that*

$$\phi_A(x) \in T\{\phi_A(\Gamma_A x)\}, \qquad \textit{for all } x \in A,$$

then (G, Γ) *admits a function* ϕ *such that*

$$\phi(x) \in T\{\phi(\Gamma x)\}, \qquad \textit{for all } x \in G.$$

PROOF. Since there is no interaction between connected components, we may take G to be connected (hence countable, because of local finiteness [1, p. 18]). Let $G = \{x_1, x_2, \cdots\}$ and define a sequence of subsets of G by setting $G_n = \{x_1, x_2, \cdots, x_{p_n}\}$, where p_1 is the least integer r such that $r > 1$ and $\{x_1, x_2, \cdots, x_r\}$ contains Γx_1, and p_n, for $n > 1$, is the least integer r such that $r > p_{n-1}$ and $\{x_1, x_2, \cdots, x_r\}$ contains $\Gamma x_1 \cup \Gamma x_2 \cup \cdots \cup \Gamma x_n$. For each n, let A_n be the set of all mappings $\phi: G_n \to K$ such that $\phi(x_i) \in T\{\phi(\Gamma x_i)\}$, for $i = 1, 2, \cdots, n$. Each A_n is nonempty since it contains ϕ_{G_n}, which exists by hypothesis. Moreover, each A_n is finite and A_n and A_m are disjoint for $n \neq m$, since $G_n \neq G_m$. Define a relation between mappings in A_{n-1} and A_n by setting $\phi < \psi$ if and only if $\psi|_{G_{n-1}} = \phi$, where $\psi|_{G_{n-1}}$ is the restriction of ψ to G_{n-1}. Then, for each $\psi \in A_n$, there is a $\phi \in A_{n-1}$ with $\phi < \psi$, in fact, $\phi = \psi|_{G_{n-1}}$ will do. Thus, by König's result, there is a sequence $\phi_1 < \phi_2 < \cdots < \phi_n < \cdots$, with $\phi_n \in A_n$, for all n. Now define a ϕ on all of G by $\phi(x_n) = \phi_n(x_n)$, for all $x_n \in G$. It is immediately verified that ϕ has the required property.

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[1] A graph (G, Γ), where Γ maps elements of G into subsets of G, is called locally finite if Γx and $\Gamma^{-1}x$ are finite, for every $x \in G$.

REFERENCES

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2. R. Rado, *Axiomatic treatment of rank in infinite sets*, Canad. J. Math. 1 (1949), 337–343.

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CONCERNING CONTINUOUS IMAGES OF COMPACT ORDERED SPACES

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It is the purpose of this paper to prove that if each of X and Y is a compact Hausdorff space containing infinitely many points, and $X\times Y$ is the continuous image of a compact ordered space L, then both X and Y are metrizable.[2] The preceding theorem is a generalization of a theorem [1] by Mardešić and Papić, who assume that X, Y, and L are also connected. Young, in [3], shows that the Cartesian product of a "long" interval and a real interval is not the continuous image of any compact ordered space.

In this paper, the word compact is used in the "finite cover" sense. The phrase "ordered space" means a totally ordered topological space with the order topology. A subset M of a topological space is said to be heriditarily separable provided each subset of M is separable. If a and b are points of an ordered space L and $a<b$, then $[a, b]$ ((a, b)) will denote the set of all points x of L such that $a\leqq x\leqq b$ ($a<x<b$), provided there is one; also, $[a, b]$ will be used even in the case where $a=b$. A subset M of an ordered space L is convex provided that if $a\in M$, $b\in M$, and $a<b$, then $[a, b]\subset M$. If M is a subset of an ordered space L, then $G(M)$ will denote the set of all ordered pairs (a, b) such that (1) $a\in M$, $b\in M$, and $a<b$, and (2) $\{a, b\}=M\cdot[a, b]$, provided there is one.

LEMMA 0. *If M is a compact subset of the ordered space L, then the relative topology of L on M is the same as the order topology on M.*

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[2] The referee has informed the author that the theorem of this paper was proved independently by A. J. Ward (Cambridge, England).