HOMOTOPY GROUPS OF COMPACT ABELIAN GROUPS

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The object of this paper is to prove the following:

THEOREM. *For any compact Abelian group G, $\pi_n(G)=0$ for $n\geqq 2$ and $\pi_1(G)$ is isomorphic to the group of homomorphisms of G^* (the Pontrjagin dual of G) into Z (the group of integers).*

Preliminaries. B_n (resp. S_{n-1}) will denote the subset of R^n consisting of those x such that $\|x\|\leqq 1$ (resp. $\|x\|=1$). x_0 will denote the point $(1, 0, 0, \cdots, 0)$ of R^n and T will denote S_1 made into a topological group by using complex multiplication. All groups will be assumed to be Abelian.

For a based topological space X and a topological group G, let $\mathcal{C}(X, G)$ denote the set of maps (i.e., base point preserving continuous maps) of X into G (where G is considered to be a based topological space with base point 0). In an obvious fashion, $\mathcal{C}(X, G)$ can be endowed with a group structure. In case X is also a topological group (considered as a based topological space with base point 0) let $\mathrm{Hom}(X, G)$ denote the subgroup of $\mathcal{C}(X, G)$ consisting of those maps which are homomorphisms. The spaces B_n and S_{n-1} will all be assumed to be based with x_0 as a base point.

If G is a discrete or compact group we let G^* denote the group $\mathrm{Hom}(G, T)$ endowed with the topology of compact convergence. If G is compact (resp. discrete) then it is known that G^* is discrete (resp. compact). Also for two discrete or two compact groups G_1 and G_2 we have an isomorphism from the group $\mathrm{Hom}(G_1, G_2)$ onto $\mathrm{Hom}(G_2^*, G_1^*)$ where an element $f\in\mathrm{Hom}(G_1, G_2)$ is mapped onto its transpose f^*. Furthermore a sequence of compact (or discrete) groups

$$G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3$$

is exact if and only if the sequence

$$G_3^* \xrightarrow{g^*} G_2^* \xrightarrow{f^*} G_1^*$$

is exact (see Weyl [1]).

Using a general existence proof (see Bourbaki [2, p. 44, Theorem CST 22]), it can be shown that for every based topological space X

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there is a compact group $\tilde{X}$ and a map $i: X \to \tilde{X}$ such that if $\phi: X \to G$ is any map where G is a compact group, then there is a unique map $f: \tilde{X} \to G$ such that $f \circ i = \phi$. Thus for any compact group G we get a natural isomorphism between the groups $\mathrm{Hom}(\tilde{X}, G)$ and $\mathcal{C}(X, G)$. We say that i is a free compact group on X. It is easy to see that i is an injection if X is completely regular since, if G is taken to be the product of sufficiently many copies of T, then ϕ can be chosen to be an injection.

Now let $j_n: S_n \to \tilde{S}_n$ and $k_n: B_n \to \tilde{B}_n$ be free compact groups on S_n and B_n, respectively, and let h_n be the unique continuous homomorphism from $\tilde{S}_n$ into $\tilde{B}_n$ such that

$$\begin{array}{ccc} S_n & \xrightarrow{i_n} & B_{n+1} \\ \downarrow j_n & & \downarrow k_n \\ \tilde{S}_n & \xrightarrow{h_n} & \tilde{B}_{n+1} \end{array}$$

is commutative where i_n is the canonical injection.

Now for any topological group G, $\pi_n(G)$ is isomorphic to the quotient of the group $\mathcal{C}(S_n, G)$ by the subgroup of $\mathcal{C}(S_n, G)$ consisting of those elements that can be continuously extended to B_{n+1} (see Hu [**3**, p. 139, Example G]). But this subgroup is precisely the image of $\mathcal{C}(B_{n+1}, G)$ in $\mathcal{C}(S_n, G)$ under the restriction map. Thus $\pi_n(G)$ equals the cokernel of the restriction map

$$\mathcal{C}(B_{n+1}, G) \to \mathcal{C}(S_n, G).$$

But we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B_{n+1}, G) & \xrightarrow{\qquad} & \mathcal{C}(S_n, G) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(\tilde{B}_{n+1}, G) & \xrightarrow{\mathrm{Hom}(h_n, G)} & \mathrm{Hom}(\tilde{S}_n, G) \end{array}$$

where the vertical maps are the natural isomorphisms and the top map is the restriction map. Thus for any compact group G, $\pi_n(G)$ and $\mathrm{Coker}(\mathrm{Hom}(h_n, G))$ are isomorphic. But

$$\begin{array}{ccc} \mathrm{Hom}(\tilde{B}_{n+1}, G) & \xrightarrow{\mathrm{Hom}(h_n, G)} & \mathrm{Hom}(S_n, G) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(G^*, \tilde{B}_{n+1}^*) & \xrightarrow{\mathrm{Hom}(G^*, h_n^*)} & \mathrm{Hom}(G^*, \tilde{S}_n^*) \end{array}$$

is commutative where the vertical mappings are the transpose mappings (hence are isomorphisms). Thus $\pi_n(G)$ is isomorphic to

Coker(Hom(G^*, h_n^*)). We will try to show that Coker(Hom(G^*, h_n^*)) is isomorphic to Hom(G^*, Coker h_n^*)

PROOF OF THEOREM. Let

$$0 \rightarrow M_n \xrightarrow{t_n} \tilde{S}_n \xrightarrow{h_n} \tilde{B}_{n+1} \xrightarrow{p_n} N_n \rightarrow 0$$

be an exact sequence of compact groups. Then

$$0 \longleftarrow M_n^* \xleftarrow{t_n^*} \tilde{S}_n^* \xleftarrow{h_n^*} \tilde{B}_{n+1}^* \xleftarrow{p_n^*} N_n^* \longleftarrow 0$$

is an exact sequence of discrete groups. Thus M_n^* is isomorphic to Coker h_n^*. But

$$\begin{array}{ccc} S_n^* & \xleftarrow{\quad h_n^* \quad} & \tilde{B}_{n+1} \\ \downarrow & & \downarrow \\ \mathcal{C}(S_n, T) & \longleftarrow & \mathcal{C}(B_{n+1}, T) \end{array}$$

is commutative with the vertical maps, the natural isomorphisms and the bottom map, the restriction map. But then Coker h_n^* is isomorphic to $\pi_n(T)$ since the cokernel of the bottom map is $\pi_n(T)$. Thus $M_1^* \cong Z$ and $M_n^* = 0$ for $n \geqq 2$ since $\pi_1(T) = Z$ and $\pi_n(T) = 0$ for $n \geqq 2$.

Furthermore we claim N_n^* is divisible for $n \geqq 1$. For N_n^* is isomorphic to the kernel of the map

$$\tilde{B}_{n+1}^* \xrightarrow{h_n^*} \tilde{S}_n^*,$$

hence to the kernel H of the restriction map

$$\mathcal{C}(B_{n-1}, T) \rightarrow \mathcal{C}(S_n, T).$$

Let B_{n+1}/S_n be the quotient space of B_{n+1} by the equivalence relation which identifies all the points in S_n. Then H is isomorphic to $\mathcal{C}(B_{n+1}/S_n, T)$. But B_{n+1}/S_n is homeomorphic to S_{n+1}. So H is isomorphic to $\mathcal{C}(S_{n+1}, T)$. But $\pi_{n+1}(T) = 0$ for $n \geqq 1$, hence any map $S_{n+1} \rightarrow T$ is homotopic to a constant mapping. Thus, the map $S_{n+1} \rightarrow T$ can be "factored through R" by the exponential map $R \rightarrow T$ (see Dieudonné [4, p. 248]). Thus $\mathcal{C}(S_{n+1}, T)$ is isomorphic to a quotient of $\mathcal{C}(S_{n+1}, R)$ which is clearly divisible. Thus $\mathcal{C}(S_{n+1}, T)$ is divisible and so H and N_n^* are divisible.

Recalling that $M_n^* = Z$ or 0 we see that the sequence

$$0 \longleftarrow M_n^* \xleftarrow{t_n^*} S_n^* \xleftarrow{h_n^*} B_{n+1}^* \xleftarrow{p_n^*} N_n^* \longleftarrow 0$$

is split exact, i.e., it is exact and the kernel of each map is a direct summand of the domain of the map. Hence, for a compact group G,

$$0 \longleftarrow \mathrm{Hom}(G^*, M_n^*) \xleftarrow{\mathrm{Hom}(G^*, t_n^*)} \mathrm{Hom}(G^*, \tilde{S}_n^*) \xleftarrow{\mathrm{Hom}(G^*, h_n^*)}$$
$$\longleftarrow \mathrm{Hom}(G^*, \tilde{B}_{n+1}^*) \xleftarrow{\mathrm{Hom}(G^*, p_n^*)} \mathrm{Hom}(G^*, N_n^*) \longleftarrow 0$$

is exact (even split exact). Thus $\mathrm{Coker}(\mathrm{Hom}(G^*, h_n^*))$ is isomorphic to $\mathrm{Hom}(G^*, M_n^*)$. But $\mathrm{Coker}(\mathrm{Hom}(G^*, h_n^*))$ is isomorphic to $\pi_n(G)$. Hence $\pi_n(G)=0$ for $n \geqq 2$ since $M_n^*=0$ for $n \geqq 2$ and $\pi_1(G)$ is isomorphic to $\mathrm{Hom}(G^*, Z)$ since $M_1^* \cong Z$. This completes the proof.

Bibliography

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