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ON SOME GEOMETRIC INEQUALITIES

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1. Let C be a closed curve of class C^2 in Euclidean n-space E_n. We write the equation of C as $\mathbf{x}=\mathbf{x}(s)$, $0 \leqq s \leqq L(C)$, where s denotes arc length and $L(C)$ is the length of C. Denoting differentiation with respect to s by a dot, we define the *total curvature* of C as

$$K(C) = \int_C |\ddot{\mathbf{x}}|\, ds. \tag{1}$$

It is proved in [1] that if C is constrained to lie in a ball of radius r, then

$$L(C) \leqq rK(C). \tag{2}$$

This result is a slight sharpening of an inequality of I. Fáry [2]. The proof given in [1] depends on an integralgeometric lemma for the 2-dimensional case, together with a reduction of the n-dimensional to the 2-dimensional case by developing the curve into a plane. The proof yields no information about curves for which equality occurs in (2).

In §2 we give a simple, direct proof of (2) and characterize those curves for which equality holds. We also obtain a sharpening of an inequality of Rešetnjak [3]. A generalization to surfaces is considered in §3.

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2. Let C be contained in the ball $|\mathbf{x}| \leqq r$. Then

$$
\begin{aligned}
L(C) &= \int_C \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, ds = \mathbf{x}(s) \cdot \dot{\mathbf{x}}(s) \Big|_{s=0}^{s=L(C)} - \int_C \mathbf{x} \cdot \ddot{\mathbf{x}}\, ds \\
&= - \int_C \mathbf{x} \cdot \ddot{\mathbf{x}}\, ds \leqq \int_C |\mathbf{x} \cdot \ddot{\mathbf{x}}|\, ds \leqq \int_C |\mathbf{x}|\, |\ddot{\mathbf{x}}|\, ds \qquad (3) \\
&\leqq r \int_C |\ddot{\mathbf{x}}|\, ds = rK(C),
\end{aligned}
$$

establishing (2).

Equality holds throughout if, and only if,

$$|\mathbf{x}(s)| = r, \tag{4a}$$

$$\lambda(s)\mathbf{x}(s) = \ddot{\mathbf{x}}(s), \tag{4b}$$

for $0 \leqq s \leqq L(C)$, where $\lambda = \lambda(s)$ is a scalar function. (4a) and (4b) imply that $\mathbf{x}(s)$ is a solution of the differential equation

$$\ddot{\mathbf{x}} + r^{-2}\mathbf{x} = 0. \tag{5}$$

If we specify $\mathbf{x}(0) = (r, 0, 0, \cdots, 0)$, $\dot{\mathbf{x}}(0) = (0, 1, 0, \cdots, 0)$, the solution of (5) is uniquely determined, and indeed is proved by

$$\mathbf{x}(s) = r(\cos r^{-1}s, \sin r^{-1}s, 0, \cdots, 0). \tag{6}$$

Since C is closed, we have $0 \leqq s \leqq 2n\pi r$, for some positive integer n.

Thus, those curves for which equality holds in (2) are circles of radius r traversed a certain number of times. (One also sees this by observing that (4a) implies C is a spherical curve, while (4b) is the condition that C be a geodesic on the sphere.)

If C is not necessarily closed and has diameter d, then C can be enclosed by a sphere of radius $r = d\sqrt{(n/(2n+2))}$, by a theorem of Jung (see [**4**, p. 78]). It then follows from the first line of (3) that

$$L(C) \leqq 2r + rK(C) = d\sqrt{\left(\frac{n}{2n+2}\right)}\,[2 + K(C)]. \tag{7}$$

For $n = 3$, (7) gives,

$$L(C) \leqq d\sqrt{\left(\frac{3}{8}\right)}\,[2 + K(C)], \tag{8}$$

a sharpening of [**3**, Theorem 3].

For rectifiable curves, $K(C)$ is defined by

$$K(C) = \sup K(P), \tag{9}$$

where the supremum is taken over all polygons P inscribed in C, $K(P)$ being the sum of the "exterior" angles at the vertices of P (see [5]). Any polygon can be approximated by a curve of class C^2, having the same total curvature and slightly smaller length (by "rounding off" the vertices). This observation enables one to establish (2) and (7) for rectifiable curves, using the result for C^2 curves.

3. It is proved in [6] that if S is a compact, orientable $(n-1)$-dimensional manifold of class C^2 imbedded in E_n, then

$$\int_S M_r dA + \int_S p M_{r+1} dA = 0, \qquad r = 0, \cdots, n-2, \tag{10}$$

where M_r is the rth elementary symmetric function of the principal curvatures $k_1, \cdots, k_{n-1}$, of S, divided by the number of terms, and p is the support function of S. The case $r=0$ gives

$$A(S) = -\int_S p M_1 dA, \tag{11}$$

where $(n-1)M_1 = k_1 + \cdots + k_{n-1}$, and $A(S)$ is the area of S. In particular, if S is contained in a ball of radius r, we have an analogue of (2):

$$A(S) \leqq r \int_S |M_1| dA, \tag{12}$$

and equality holds if, and only if, S lies on a sphere of radius r.

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