A COMPLEX-VARIABLES PROOF OF HÖLDER'S INEQUALITY

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Hölder's inequality says that if $f, g \geqq 0$, if $p > 1$, and if $1/p + 1/q = 1$, then

$$\int fg \leqq \left\{ \int f^p \right\}^{1/p} \left\{ \int g^q \right\}^{1/q}. \tag{1}$$

We have deliberately omitted the measure with respect to which integration is performed, and measurability and boundedness hypotheses on the functions f and g. The reader can supply his own virtually at will. For example, $\int f$ could mean $\int_0^1 f(x)\, dx$, where dx is Lebegsue measure and f and g are assumed to be bounded and Lebesgue measurable.

Replacing f^p by f and g^q by g, and writing $1/q = s$, we must prove that

$$\phi(s) \leqq 1 \quad \text{for} \quad 0 < s < 1, \tag{2}$$

where

$$\phi(s) = \int f^{1-s} g^s \Big/ \left\{ \int f \right\}^{1-s} \left\{ \int g \right\}^s. \tag{3}$$

But now let $s = \sigma + it$. We see that $\phi(s)$ is defined and continuous in the strip $0 \leqq \sigma \leqq 1$, and since we can differentiate "by hand", we see that ϕ is analytic in the interior of the strip. It is easy to see that $|\phi(\sigma+it)| \leqq \phi(\sigma)$ for $0 \leqq \sigma \leqq 1$, because $|\int f^{1-s} g^s| \leqq \int f^{1-\sigma} g^\sigma$, $|\{\int f\}^{1-s}| = \{\int f\}^{1-\sigma}$, and $|\{\int g\}^s| = \{\int g\}^\sigma$. From the maximum modulus principle, then $|\phi(\sigma+it)| \leqq \max \{|\phi(0)|, |\phi(1)|\} = 1$, and the result is proved. We emphasize that only the local maximum modulus principle was used above.

It is also easy to show by this method that if equality holds in (1), or, equivalently, in (2), then f is proportional to g almost everywhere. For the case $s = 1/2$, we have the standard proof that the quadratic function $u(\lambda) = \int (f - \lambda g)^2$ has vanishing discriminant, and therefore a root λ_0, so that $f = \lambda_0 g$ almost everywhere. The general case reduces to this case by the maximum modulus principle. For if $|\phi(s)| = 1$ for some $s = \sigma + it$ with $0 < \sigma < 1$, then ϕ is a constant, and hence $\phi(1/2) = 1$.

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