INTEGER TOPOLOGIES[1]

LOWELL A. HINRICHS

Let $(I, +, \cdot)$ be a ring. A central problem in topological algebra is to determine what (Hausdorff) topologies on I are compatible with the ring operations. If $\mathcal{I}$ is any family, of ideals, with the finite intersection property and such that $\bigcap \mathcal{I} = (0)$, then $\mathcal{I}$ is a basis for the neighborhood system of zero with respect to a ring topology on I. Such a topology is called an *ideal topology*. If $(I, +, \cdot)$ is the ring of integers there are, of course, many such topologies. Indeed, a natural question is whether or not all ring topologies on the integers are of this form.[2] In this paper we answer this question in the negative.

In §1 a method is given for constructing nonideal topologies on the integers. In §2 it is shown that there exist uncountably many such topologies. Finally, in §3, these methods are utilized to demonstrate the existence of a ring topology on I which is not first countable.

1. **Nonideal topologies.**

Throughout the paper, I will denote the ring of integers, N the set of non-negative integers, and N_r the set of non-negative integers less than or equal to r. If $A \subseteq I$, we will denote $\{|a|, a \in A\}$ by $|A|$.

LEMMA 1. *Suppose $(I_n^m)_{n \leq m}$ is a double sequence of finite subsets of I containing zero and such that for all n, $m \in N$ with $n+1 \leqq m$,*

$$I_{n+1}^m + \bigcup_{r=n+1}^{m} I_{n+1}^r \subseteq \bigcup_{s=n}^{m} I_n^s, \tag{1}$$

$$I_{n+1}^m \cdot \bigcup_{r=n+1}^{m} I_{n+1}^r \subseteq \bigcup_{s=n}^{m} I_n^s, \tag{2}$$

$$N_{n+1} \cdot I_{n+1}^m \subseteq \bigcup_{s=n}^{m} I_n^s, \tag{3}$$

$$I_{n+1}^m \subseteq I_n^m, \tag{4}$$

$$I_n^m = -I_n^m, \tag{5}$$

$$\inf \left| I_0^{n+1} \sim \{0\} \right| - \sup \left| I_0^n \right| \geqq n+1. \tag{6}$$

Presented to the Society, January 26, 1964 under the title *Nonideal topologies on the ring of integers*; received by the editors July 17, 1963.

[1] This research was supported in part by the National Science Foundation grant NSF-GP-1913.

[2] The question of the existence of such topologies was posed by Seth Warner.

Let $\mathcal{V} = \{V_n: V_n = \bigcup_{m=n}^{\infty} I_n^m\}$, $n \in N$. *Then* $\mathcal{V}$ *is a fundamental system of neighborhoods of zero with respect to a nonideal ring topology on* I.

PROOF. It is apparent that $\mathcal{V}$ is a decreasing chain of subsets containing zero and hence is a filter base. That $\mathcal{V}$ defines a ring topology follows from the fact that $V_{n+1} \pm V_{n+1} \subseteq V_n$, $V_{n+1} \cdot V_{n+1} \subseteq V_n$, and $n \cdot V_{n+1} \subseteq V_n$. From (4), (6) and the definition of V_n, it follows that $\inf |V_n \sim \{0\}| = \inf |I_0^n \sim \{0\}| \geqq n+1$. Thus $\bigcap \mathcal{V} = \{0\}$ and hence the topology is Hausdorff. To see that $\mathcal{V}$ does not define an ideal topology, let pI be a nonzero ideal of I. By (6) $\inf |I_0^{p+1} \sim \{0\}| - \sup |I_0^p| \geqq p+1$. Since $\sup |I_0^p|$ and $\inf |I_0^{p+1} \sim \{0\}|$ are consecutive elements of V_0, $pI \not\subseteq V_0$. Consequently, no member of $\mathcal{V}$ contains a nonzero ideal.

LEMMA 2. *There exists a double sequence* $(I_n^m)_{n \leq m}$, *of subsets of* I, *having properties* (1)–(6).

PROOF. Using induction and the countable axiom of choice, we will construct the desired double sequence. The reader will observe that the necessity of invoking the axiom of choice can easily be avoided; however, its use at this stage simplifies the proof of later theorems.

We begin the construction by setting $I_0^0 = \{0\}$. The inductive assumption is that we have defined a finite double sequence $(I_n^m)_{0 \leq n \leq m \leq k}$, satisfying (1)–(6).

Next we choose $a_{k+1} > (k+2) \sup |I_0^k|$ and set

$$I_{k+1}^{k+1} = J_{k+1} = \{0, a_{k+1}, -a_{k+1}\}$$

and

$$I_k^{k+1} = J_k = (J_{k+1} + J_{k+1}) \cup (J_{k+1} \cdot J_{k+1}) \cup (N_k \cdot J_{k+1}).$$

For $0 \leqq r < k$ we set

$$J_r = (J_{r+1} + J_{r+1}) \cup (J_{r+1} \cdot J_{r+1}) \cup (N_r \cdot J_{r+1}) \cup \left[J_{r+1} + \bigcup_{s=r+1}^{k} I_{r+1}^s\right] \cup \left[J_{r+1} \cdot \bigcup_{s=r+1}^{k} I_{r+1}^s\right].$$

Finally for $0 \leqq r < k$, we set

$$I_r^{k+1} = \left(J_r \sim \bigcup_{s=r}^{k} I_r^s\right) \cup \{0\}.$$

Before proceeding with the inductive step, it is convenient to establish the following.

LEMMA 3. *Every nonzero element of* I_r^{k+1} *is of the form* $pa_{k+1}+b$, *where* $p\neq 0$ *and* $b\in\bigcup_{s=r}^k I_r^s$.

PROOF. We suppose that the lemma is false and that r is the largest integer such that the result fails. From the definition of I_{k+1}^{k+1} and I_k^{k+1}, it is clear that $r<k$. Let c be a nonzero element of I_r^{k+1}, which is not of the desired form. We will obtain a contradiction by assuming that c is an element of one of the sets whose union is J_r.

If $c\in J_{r+1}+J_{r+1}$, then $c=a+b$ where a, $b\in J_{r+1}$. First we will show that any element e of J_{r+1} can be written in the form $e=fa_{k+1}+u$ where $f\in I$ and $u\in\bigcup_{s=r+1}^k I_{r+1}^s$. It is evident from the definitions that for $r+1\leqq k$ we have $J_{r+1}\subseteq I_{r+1}^{k+1}\cup(\bigcup_{s=r+1}^k I_{r+1}^s)$. If $e\in I_{r+1}^{k+1}$, the assertion follows from the maximality of r; if $e\in\bigcup_{s=r+1}^k I_{r+1}^s$ we can choose $f=0$, $u=e$ and again the assertion holds. Thus we may choose f, $g\in I$ and u, $v\in\bigcup_{s=r+1}^k I_{r+1}^s$ such that $a=fa_{k+1}+u$ and $b=ga_{k+1}+v$. Then $c=a+b$ $=(f+g)a_{k+1}+(u+v)$. Moreover, $u+v\in\bigcup_{s=r}^k I_r^s$, since $u+v$ $\in(\bigcup_{s=r+1}^k I_{r+1}^s)+(\bigcup_{s=r+1}^k I_{r+1}^s)\subseteq\bigcup_{s=r+1}^k(I_{r+1}^s+\bigcup_{t=r+1}^s I_{r+1}^t)\subseteq\bigcup_{s=r+1}^k\bigcup_{t=r}^s I_r^t$ $=\bigcup_{s=r}^k I_r^s$. Thus by our choice of c, $p+q=0$ and hence $0\neq c=u+v$ $\in\bigcup_{s=r}^k I_r^s$. From the definition of I_r^{k+1} it follows that $c\notin I_r^{k+1}$, contrary to our initial assumption. A similar argument shows that c is not an element of any of the other defining sets whose union is J_r. This establishes the lemma.

Next we will show that the family $(I_n^m)_{n\leq m\leq k}\cup(I_r^{k+1})_{0\leq r\leq k+1}$ satisfies (1)–(6). First we will verify that the family satisfies (1).

From the definitions we have that J_r contains $(J_{r+1}+J_{r+1})$ and $(J_{r+1}+\bigcup_{s=r+1}^k I_{r+1}^s)$, and that J_{r+1} contains I_{r+1}^{k+1}. Thus $J_r\supseteq(J_{r+1}+I_{r+1}^{k+1})$ $\cup$ $(J_{r+1}+\bigcup_{s=r+1}^k I_{r+1}^s)$ $=$ $J_{r+1}+\bigcup_{s=r+1}^k I_{r+1}^s$. Moreover, since I_r^{k+1} $=(J_r\sim\bigcup_{s=r+1}^k I_{r+1}^s)\cup\{0\}$, we have $I_r^{k+1}\supseteq(J_{r+1}+\bigcup_{s=r+1}^{k+1} I_{r+1}^s)\sim\bigcup_{s=r}^k I_r^s$ $\supseteq(I_{r+1}^{k+1}+\bigcup_{s=r+1}^{k+1} I_{r+1}^s)\sim\bigcup_{s=r+1}^k I_r^s$ and hence $\bigcup_{s=r}^{k+1} I_r^s\supseteq I_{r+1}^{k+1}+\bigcup_{s=r+1}^{k+1} I_{r+1}^s$. This establishes (1); a similar argument establishes the validity of (2) and (3).

If a is a nonzero element of I_{r+1}^{k+1}, then, by Lemma 3, $a=pa_{k+1}+b$, where $p\neq 0$ and $b\in\bigcup_{s=r+1}^k I_{r+1}^s$. Thus $|a|=|pa_{k+1}+b|\geqq|p|\cdot|a_{k+1}|$ $-|b|\geqq|a_{k+1}|-\sup|I_0^k|\geqq(k+2)\sup|I_0^k|-\sup|I_0^k|\geqq(k+1)\sup|I_0^k|$. Hence $a\in J_r\sim\bigcup_{s=r}^k I_r^s=I_r^{k+1}$ and (4) follows. Condition (5) follows from the fact that all of the defining sets of J_r are symmetric as is $\bigcup_{s=r}^k I_r^s$. Finally, from the verification of (4) it is clear that $\inf|I_0^{k+1}\sim\{0\}|-\sup|I_0^k|\geqq k+1$.

This completes the induction. Lemma 2 now follows from the obvious application of Zorn's lemma. Combining Lemmas 1 and 2, we now have:

THEOREM. *There exists a nonideal ring topology on the ring of integers.*

For lack of a better name we will call topologies obtained as above *topologies of type* (*). The sequence $a_0=0$, $a_n=\inf|I_n^n\sim\{0\}|$, $n>0$, will be called the *defining sequence* (*of the topology*).

2. **Uncountably many ring topologies.** In this section it is shown that there exist uncountably many ring topologies on I.

Let $\{\mathfrak{V}(n)\}_{n\in N}$ be a sequence (possibly finite) of fundamental neighborhood systems of zero with respect to topologies of type (*). We will denote the defining sequence of $\mathfrak{V}(n)$ by $(a(n)_i)$. The sets $I(n)_r^s$ and $V(n)_r$ will have the obvious interpretations.

LEMMA 4. *Let $\mathfrak{V}(n)$ be a sequence (possibly finite) of fundamental neighborhood systems of zero with respect to topologies of type* (*). *Then there exists a defining sequence $(a_i)_{i\in N}$ such that for every $0\leqq t\leqq s$ there exists a $q\in N$ such that*

$$\text{(A)}\qquad \sup|I_0^s|<\inf|I(t)_0^q\sim\{0\}|<\sup|I(t)_0^q|<\inf|I_0^{s+1}\sim\{0\}|.$$

PROOF. We will inductively construct the sequence (a_i). Clearly we can choose a_1 so that $\sup|I(0)_0^1|<\inf|I_0^1\sim\{0\}|$. Suppose that for $0\leqq s+1\leqq k$ we have defined a_i's such that (A) is satisfied. It is clear that, for each $0\leqq t\leqq k+1$, there exists $m\in N$ such that $\sup|I_0^k|<\inf|I(t)_0^m\sim\{0\}|$. For each $0\leqq t\leqq k+1$, let m_t be the first such.

Since $\bigcup_{t=0}^{k+1} I(t)_0^{m_t}$ is finite the supremum of this set exists. Choose $a_{k+1}>3(k+1)\sup|\bigcup_{t=0}^{k+1} I(t)_0^{m_t}|$. As in the proof of Lemma 2, it follows that

$$\sup\left|\bigcup_{t=0}^{k+1} I(t)_0^{m_t}\right|<\inf|I_0^{k+1}\sim\{0\}|.$$

Hence we have

$$\sup|I_0^k|<\inf|I(t)_0^{m_t}\sim\{0\}|<\sup|I(t)_0^{m_t}|<\inf|I_0^{k+1}\sim\{0\}|.$$

The lemma now follows by the usual Zorn's lemma argument.

COROLLARY. *There exist uncountably many topologies of type* (*).

PROOF. Suppose there exist at most countably many and let $\mathfrak{V}(n)$ be the corresponding bases for the neighborhood systems of zero. By Lemma 4, there exists a defining sequence which yields a topology not in the list.

3. **Ring topologies which are not first countable.**

LEMMA 5. *If $\mathfrak{I}$ is a topology of type* (*), *there exists a topology $\mathfrak{I}^*$ of type* (*) *such that $\mathfrak{I}\subsetneqq\mathfrak{I}^*$.*

PROOF. Let the fundamental neighborhood system of zero with respect to $\mathfrak{I}$ be denoted by $\mathfrak{V}(i)$. In the construction employed in the proof of Lemma 4, we choose $a_{k+1} \in V(1)_{k+1}$. The resulting topology $\mathfrak{I}^*$ is strictly finer than $\mathfrak{I}$.

LEMMA 6. *If $\mathfrak{I}$ is a nondiscrete, first countable ring topology for I, then there exists a topology $\mathfrak{I}^*$ of type (*) such that $\mathfrak{I} \subsetneq \mathfrak{I}^*$.*

PROOF. It is easily seen that there exists a fundamental system $\{U_n; n \in N\}$ of $\mathfrak{I}$-neighborhoods of zero such that for all $n \in N$

(i) $U_{n+1} + U_{n+1} \subseteq U_n$,

(ii) $U_{n+1} \cdot U_{n+1} \subseteq U_n$,

(iii) $N_{n+1} \cdot U_{n+1} \subseteq U_n$,

(iv) $U_n = -U_n$.

Clearly we may again apply the method of Lemma 4 to choose a defining sequence $(a_i)_{i \in N}$ with the additional property that $a_i \in U_i$. It follows that the topology $\mathfrak{I}'$, of type (*), so defined is finer than $\mathfrak{I}$. By Lemma 5, there exists a topology $\mathfrak{I}^*$ of type (*) strictly finer than $\mathfrak{I}$.

COROLLARY. *There exists a ring topology on I which is not first countable.*

PROOF. The standard Zorn's lemma argument shows that any fundamental neighborhood system of zero, with respect to a nondiscrete ring topology on I, is contained in a maximal system of the same type. Thus, there exists a maximal nondiscrete ring topology on I. In view of the preceding corollary, such topologies are not first countable.

DUKE UNIVERSITY