

ON FIXED POINTS OF THE COMPOSITE OF COMMUTING FUNCTIONS

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1. Introduction. Let $f(t)$ and $g(t)$ be continuous functions mapping $[0, 1]$ into $[0, 1]$ which commute under substitution, i.e., $f(g(t)) = g(f(t))$. In 1954, E. Dyer "conjectured"² that $f(t)$ and $g(t)$ must have a common fixed point in $[0, 1]$, $t_0 = f(t_0) = g(t_0)$ for some t_0 in $[0, 1]$. For certain special functions $f(t)$ and $g(t)$ the conjecture has been verified, but in general the question of existence of this common fixed point remains open.

An equivalent way of phrasing the conjecture is to say that $f(t)$ and $h(t) = f(g(t))$ must have a common fixed point. In this connection, it is known that $f(t)$ and $g(t)$ "permute" the fixed points of $h(t)$. In particular, if $h(t)$ has a finite number of fixed points t_1, \dots, t_n , then $f(t_1), \dots, f(t_n)$ is a permutation of t_1, \dots, t_n . (This follows from the fact that (1) $f(t)$ and $h(t)$ commute and (2) $f(t_i) = f(t_j)$ implies $t_i = t_j$ by applying $g(\cdot)$ to both sides.) In this note, we investigate more closely the permutation just mentioned for the case in which $h(t)$ has a finite number of fixed points.

We begin with the observation that any fixed point t_0 of $h(t)$ is one of three types. Type I, *up-crossing*: $h(t)$ passes from below to above the diagonal as t increases through t_0 . Type II, *down-crossing*: $h(t)$ passes from above to below the diagonal as t increases through t_0 . Type III, *touching*: $h(t)$ does not cross the diagonal at t_0 . Included in Type II will be (1) $t = 1$ if $h(1) = 1$ and $h(t) > t$ near $t = 1$, and (2) $t = 0$ if $h(0) = 0$ and $h(t) < t$ near $t = 0$. Included in Type III will be (1) $t = 1$ if $h(1) = 1$ and $h(t) < t$ near $t = 1$, and (2) $t = 0$ if $h(0) = 0$ and $h(t) > t$ near $t = 0$. We now state our main result, which indicates how $f(t)$ and $g(t)$ "preserve" the local behavior of $h(t)$ at fixed points.

THEOREM. *Let $h(t)$ have a finite number of fixed points. Then, $f(t)$ and $g(t)$ permute the fixed points of each type.*

We can use the theorem to obtain some information about Dyer's

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² Dyer raised the question in 1954. Shields asked the same question in 1955, and Dubins asked it in 1956. Isbell raised a more general question in 1957 [1].

conjecture. For any function $h(t)$, there must be one more down-crossing than up-crossing. If $h(t)$ has fewer than five fixed points, there is only one up-crossing or only one down-crossing. In either case, the permutation $\{f(t_i)\}$ has a fixed point, verifying the conjecture. The theorem gives useful information in many cases, but it gives no information about the conjecture even if $h(t)$ has only five fixed points, two up-crossings and three down-crossings.

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2. Proof of the theorem. Let $0 \leq t_1 < t_2 \cdots < t_n \leq 1$ denote the fixed points of $h(t) = f(g(t))$, and let I_k be the interval $t_k \leq t \leq t_{k+1}$. For any interval I , let $f(I)$ denote the interval into which $f(t)$ maps I . If $h(t) \geq t$ on I_k , we say I_k is an up-interval; if $h(t) \leq t$ on I_k , we say I_k is a down-interval. If two intervals I_k and I_m are both up-intervals or both down-intervals, we say they are alike. From now on, $f(g(t))$ will be denoted by simply $fg(t)$.

We establish two auxiliary lemmas.

LEMMA 1. *Let $f(t_m)$ belong to $f(I_k)$ for some t_m . Then, $t_m \geq t_k$ if I_k is an up-interval, and $t_m \leq t_{k+1}$ if I_k is a down-interval.*

For the proof, let I_k be an up-interval and let a be a point in I_k such that $f(t_m) = f(a)$. Then, $t_m = gf(t_m) = gf(a) \geq a$. The other half of the lemma is proved in a similar manner. Note, in particular, that if $f(t_k)$ and $f(t_{k+1})$ are *not* successive fixed points, then there is at least one $t_m \neq t_k, t_{k+1}$ for which $f(t_m)$ belongs to $f(I_k)$. The direction of the inequality $h(t) \leq t$ or $h(t) \geq t$ determines the location of t_m with respect to I_k . The next lemma considers the case in which $f(t_k)$ and $f(t_{k+1})$ are successive fixed points.

LEMMA 2. *Let $f(t_k)$ and $f(t_{k+1})$ be successive fixed points, say t_m and t_{m+1} . If $f(t_k) = t_m$, then I_k and I_m are alike. If $f(t_k) = t_{m+1}$, then I_k and I_m are not alike.*

To prove the first part of the lemma, we must eliminate the case in which I_k and I_m are not alike. Since $\{f(t_i)\}$ and $\{g(t_i)\}$ are inverse permutations, we can assume without loss of generality that I_k is an up-interval and I_m is a down-interval. First, we show that

$$(1) \quad f(I_k) = I_m \quad \text{and} \quad g(I_m) = I_k.$$

For suppose there were a_0 in the open interval (t_k, t_{k+1}) such that $f(a_0) = t_m$. Then, $a_0 < gf(a_0) = g(t_m) = t_k$, a contradiction. Thus, $f(a) > t_m$ for all a in (t_k, t_{k+1}) . On the other hand, if there were a_0 in the open

interval (t_k, t_{k+1}) such that $f(a_0) = t_{m+1}$, then there would exist b_0 in the interval (t_m, t_{m+1}) with $g(b_0) = a_0$, and $b_0 > fg(b_0) = f(a_0) = t_{m+1}$, a contradiction. This means that $f(I_k) = I_m$ and similarly $g(I_m) = I_k$. Next, we take any a_0 in the open interval (t_k, t_{k+1}) and define

$$(2) \quad b_0 = f(a_0), \quad a_{n+1} = g(b_n), \quad b_{n+1} = f(a_{n+1}).$$

According to (1) and to the relations $a_{n+1} = gf(a_n) > a_n$, $b_{n+1} = fg(b_n) < b_n$, it follows that $\{a_n\}$ is an increasing sequence of points in I_k approaching t_{k+1} and that $\{b_n\}$ is a decreasing sequence of points in I_m approaching t_m . By (2), we are led to the contradiction

$$(3) \quad t_{k+1} = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} g(b_n) = g(t_m) = t_k.$$

The proof of the second half of the lemma can be carried out in a similar manner.

Using the two lemmas above, one can prove the theorem. Since $\{f(t_i)\}$ and $\{g(t_i)\}$ are inverse permutations, there are exactly three cases to eliminate: (1) up-crossing at t_k and down-crossing at $f(t_k)$, (2) up-crossing at t_k and touching at $f(t_k)$, (3) down-crossing at t_k and touching at $f(t_k)$. We remark that if there is an up-crossing at t_k , then $f(t_k) = t_m$ must lie between $f(t_{k-1})$ and $f(t_{k+1})$. Because, if this were not the case, $f(I_{k-1})$ and $f(I_k)$ would have an interval I_{m-1} or I_m in common. But $gf(I_{k-1}) \subset [0, t_k]$ and $gf(I_{k+1}) \subset [t_k, 1]$, so that $gf(I_{k-1})$ and $gf(I_k)$ cannot have an interval $g(I_{m-1})$ or $g(I_m)$ in common. An immediate consequence is that $f(t_k) \neq 0, 1$ if there is an up-crossing at t_k . In fact, $f(t_k) \neq t_1, t_n$. Also, we note that if there is a down-crossing at t_k for which $f(t_k) = t_m$ does not lie between $f(t_{k-1})$ and $f(t_{k+1})$, then either t_{m-1} or t_{m+1} occurs as a value of $f(t_{k-1})$ or $f(t_{k+1})$. For example, if both $f(t_{k-1})$ and $f(t_{k+1})$ were greater than t_m , then one of them would be equal to t_{m+1} . This follows by applying Lemma 1 to the relations $t_{m+1} = f(t_j) \in f(I_{k-1})$, $t_{m+1} = f(t_j) \in f(I_k)$, to find $t_{k-1} \leq t_j \leq t_{k+1}$.

Case 1. Up-crossing at t_k and down-crossing at $t_m = f(t_k)$. In this case $t_m \neq t_1, t_n$, and I_{k-1} and I_m are down-intervals while I_k and I_{m-1} are up-intervals. We will now show that neither $f(t_{k-1}) > t_m$ nor $f(t_{k+1}) > t_m$, contradicting the statement about up-crossings in the previous paragraph. If $f(t_{k-1}) > t_m$, then $f(I_{k-1}) \supset I_m$. Since I_{k-1} is a down-interval,

$$(4) \quad [0, t_k] \supset gf(I_{k-1}) \supset g(I_m),$$

which implies that $g(I_m) \supset I_{k-1}$. This means $gf(t_{k-1}) = t_{k-1} \in g(I_m)$, which by Lemma 1 implies $f(t_{k-1}) \leq t_{m+1}$ (I_m is a down-interval). This leaves only the possibility $f(t_{k-1}) = t_{m+1}$, which by Lemma 2 implies

I_{k-1} and I_m are not alike. Contradiction. If on the other hand $f(t_{k+1}) > t_m$, then $f(I_k) \supset I_m$. Since I_k is an up-interval,

$$(5) \quad [t_k, 1] \supset gf(I_k) \supset g(I_m),$$

which implies $g(I_m) \supset I_k$. This means $gf(t_{k+1}) = t_{k+1} \in g(I_m)$, which by Lemma 1 implies $f(t_{k+1}) \leq t_{m+1}$. This leaves only the possibility $f(t_{k+1}) = t_{m+1}$, which by Lemma 2 implies I_k and I_m are alike. Contradiction.

Case 2. Up-crossing at t_k and touching at $t_m = f(t_k)$. Once again $t_m \neq t_1, t_n$. In the proof under Case 1, we showed that neither $f(t_{k-1}) > t_m$ nor $f(t_{k+1}) > t_m$ if I_m is a down-interval. By a completely analogous proof, we can show that neither $f(t_{k-1}) < t_m$ nor $f(t_{k+1}) < t_m$ if I_{m-1} is an up-interval. In case of a touching at t_m , I_{m-1} and I_m are alike, implying that either I_m is a down-interval or I_{m-1} is an up-interval. Thus, t_m cannot lie between $f(t_{k-1})$ and $f(t_{k+1})$, giving a contradiction.

Case 3. Down-crossing at t_k and touching at $t_m = f(t_k)$. Suppose first that $f(t_{k-1}) = t_{m+1}$. Lemma 2 implies that I_m is a down-interval. This means $t_m \neq t_1$ and that I_{m-1} is also a down-interval. Now, $g(I_{m-1})$ contains either I_{k-1} or I_k , so that either $gf(t_{k-1}) = t_{k-1} \in g(I_{m-1})$ or $gf(t_{k+1}) = t_{k+1} \in g(I_{m-1})$. By Lemma 1 then, either $f(t_{k-1}) \leq t_m$ or $f(t_{k+1}) \leq t_m$. The only possibility is $f(t_{k+1}) < t_m$. This in turn means $f(I_k) \supset I_{m-1}$, so that

$$(6) \quad [0, t_{k+1}] \supset gf(I_k) \supset g(I_{m-1}).$$

Thus, either $g(t_{m-1}) = t_{k+1}$ or $g(I_{m-1}) \supset I_{k-1}$. The case $g(t_{m-1}) = t_{k+1}$ is eliminated by Lemma 2 which would imply that I_{m-1} and I_k are not alike, a contradiction. On the other hand, $g(I_{m-1}) \supset I_{k-1}$ implies $gf(t_{k-1}) = t_{k-1} \in g(I_{m-1})$ which by Lemma 1 gives $f(t_{k-1}) \leq t_m$, a contradiction. The cases $f(t_{k-1}) = t_{m-1}$, $f(t_{k+1}) = t_{m-1}$, $f(t_{k+1}) = t_{m+1}$ can be eliminated in a similar manner. (It is left to the reader to show that the cases $t_k = t_1, t_n$ have been eliminated by the previous argument.) According to the remark preceding the discussion of Case 1, we have only to consider the case in which t_{m-1}, t_m , and t_{m+1} all lie between $f(t_{k-1})$ and $f(t_{k+1})$. Suppose $f(t_{k-1}) < t_{m-1}, t_m, t_{m+1} < f(t_{k+1})$. Then, $f(I_{k-1}) \supset I_{m-1}$ and $f(I_k) \supset I_m$. But

$$(7) \quad \begin{aligned} [t_{k-1}, 1] &\supset gf(I_{k-1}) \supset g(I_{m-1}), \\ [0, t_{k+1}] &\supset gf(I_k) \supset g(I_m). \end{aligned}$$

Therefore, $g(I_{m-1}) \supset I_k$ and $g(I_m) \supset I_{k-1}$. This means that $gf(t_{k+1}) = t_{k+1} \in g(I_{m-1})$ and $gf(t_{k-1}) = t_{k-1} \in g(I_m)$. Since I_m and I_{m-1} are alike, Lemma 1 states that either $f(t_{k+1}) \leq t_m$ or $f(t_{k-1}) \geq t_m$. Contradiction.

The remaining case in which $f(t_{k+1}) < t_{m-1}, t_m, t_{m+1} < f(t_{k-1})$ is handled in a similar fashion.

As a final remark, we note that in the proofs there was no essential use made of the commutativity property of $f(t)$ and $g(t)$ except at fixed points of $h(t)$.

REFERENCE

1. J. R. Isbell, *Commuting mappings of trees*, Research Problem 7, Bull. Amer. Math. Soc. **63** (1957), 419.

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