AN EXTENSION OF THE TAYLOR SUMMABILITY TRANSFORM[1]

J. P. KING

1. **Introduction.** Two matrix methods of summability which have been the object of much recent research are the Euler matrix, $E(r)=(a_{nk})$, and the Taylor matrix, $T(r)=(b_{nk})$. The elements of these matrices are defined by

$$a_{nk} = \begin{cases} \binom{n}{k} r^k(1-r)^{n-k} & (k \leqq n), \\ 0 & (k > n), \end{cases}$$

and

$$b_{nk} = \begin{cases} 0 & (k < n), \\ \binom{k}{n} r^{k-n}(1-r)^{n+1} & (k \geqq n), \end{cases}$$

respectively.

R. P. Agnew [1] has studied the matrix $E(r_n)$ which is obtained from the $E(r)$ matrix by replacing the parameter r by a parameter r_n which is dependent on n. This matrix has also been studied by Wuyts-Torfs [4], [5]. It is natural to form the matrix $T(r_n)$ from the $T(r)$ matrix in a similar manner. The elements c_{nk} of the $T(r_n)$ matrix are then defined by

$$c_{nk} = \begin{cases} 0 & (k < n), \\ \binom{k}{n} r_n^{k-n}(1-r_n)^{n+1} & (k \geqq n). \end{cases}$$

In §2 we determine necessary and sufficient conditions on r_n in order that $T(r_n)$ be regular. In §3 we determine sufficient conditions on r_n which insure that $T(r_n) \supset T(r)$, i.e., which insure that each sequence which is summable $T(r)$ is summable $T(r_n)$ to the same value.

2. **Regularity of** $T(r_n)$. A matrix $A=(a_{nk})$ is regular if and only if the well-known Silverman-Toeplitz conditions:

$$\lim_n a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots), \tag{2.1}$$

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[1] The results of this note appear in the author's doctoral dissertation written at the University of Kentucky in 1962 under the direction of Professor V. F. Cowling.

$$\lim_{n} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{2.2}$$

$$\sup_{n} \sum_{k=0}^{\infty} | a_{nk} | < \infty, \tag{2.3}$$

are satisfied.

Condition (2.1) holds trivially for the $T(r_n)$ matrix without restriction on r_n. Condition (2.2) is satisfied if and only if $|r_n| < 1$ $(n=0, 1, \cdots)$ since

$$\sum_{k=n}^{\infty} \binom{k}{n} r_n^{k-n}(1 - r_n)^{n+1} = 1 \qquad (n = 0, 1, \cdots),$$

when $|r_n| < 1$ and the series diverges if $|r_n| \geqq 1$. Furthermore, if $|r_n| < 1$,

$$\sum_{k=n}^{\infty} \binom{k}{n} | r_n |^{k-n} | 1 - r_n |^{n+1} = \left[\frac{| 1 - r_n |}{1 - | r_n |}\right]^{n+1},$$

so that (2.3) will hold if and only if

$$\left[\frac{| 1 - r_n |}{1 - | r_n |}\right]^{n+1} \leqq K \qquad (n = 0, 1, \cdots), \tag{2.4}$$

where K is a constant independent of n. That is, (2.3) will hold if and only if $\exp[(n+1) \log(|1-r_n|/(1-|r_n|))]$ is a bounded function of n. We therefore have the result:

THEOREM 2.1. *The $T(r_n)$ matrix is regular if and only if $|r_n| < 1$ $(n=0, 1, \cdots)$ and $(n+1) \log(|1-r_n|/(1-|r_n|))$ is a bounded function of n.*

The above theorem may be given the following geometrical characterization:

THEOREM 2.2. *Let M be a non-negative constant and let $k_n = \exp(M/(n+1))$ $(n=0, 1, \cdots)$. Let I_n denote the inner loop of the limaçon L_n given by*:

$$p_n = 2\,\frac{k_n^2 \cos \phi_n - k_n}{k_n^2 - 1}, \tag{2.5}$$

where (p_n, ϕ_n) are polar coordinates at $z=1$ with ϕ_n measured clockwise from the line segment joining $z=0$ and $z=1$. Then $T(r_n)$ is regular if and only if $|r_n| < 1$ and $r_n \in I_n$ $(n=0, 1, \cdots)$.

PROOF. Condition (2.4) holds if and only if

$$\frac{|1-r_n|}{1-|r_n|} \leqq e^{M/(n+1)} \qquad (n = 0, 1, \cdots),$$

where M is a non-negative constant. For fixed n this last inequality is satisfied in a region bounded by the curve L_n given by

$$|1 - r_n| = k_n(1 - |r_n|). \tag{2.6}$$

Introducing polar coordinates (p_n, ϕ_n) as described in the statement of Theorem 2.2, we have

$$1 - r_n = p_n(\cos \phi_n + i \sin \phi_n).$$

Equation (2.6) becomes

$$p_n = k_n[1 - |1 - p_n(\cos \phi_n + i \sin \phi_n)|].$$

Hence

$$(p_n - k_n)^2 = k_n^2(1 + 2k_n \cos \phi_n + p_n^2),$$

so that

$$p_n(k_n^2 - 1) - 2k_n^2 \cos \phi_n + 2k_n = 0,$$

which reduces to (2.5).

For fixed n this is the equation of a limaçon with two branches through the point $z=1$. Since $|r_n| < 1$, in order that $T(r_n)$ be regular, it is clear that r_n must lie in the inner loop I_n of L_n. This proves the theorem.

It is useful to note that the sequence of regions I_n approaches the interval $[0, 1)$ as $n \to \infty$. To see this we notice that $p_n = 2k_n/(k_n+1)$ when $\phi_n = 0$, and that this quantity tends to 1 as $n \to \infty$. It is an exercise in elementary calculus to prove that the maximum height of the boundary of I_n above the x-axis tends to 0 as $n \to \infty$.

We may note also that, in particular, $T(r_n)$ is regular if $0 \leqq r_n < 1$ $(n=0, 1, \cdots)$. Furthermore, if $r_n = r$, a constant $(n=0, 1, \cdots)$, then $T(r_n) = T(r)$ is regular if and only if $0 \leqq r < 1$. This is the known necessary and sufficient condition that $T(r)$ be regular [2].

3. **The relation** $T(r_n) \supset T(r)$. Let $\{s_i\}$ be a sequence which is $T(r)$-summable to s, where $r \neq 1$. Let $\{u_k\}$ denote the $T(r)$-transform of the sequence $\{s_i\}$ and let $\{t_k\}$ denote the $T(r_n)$-transform of the same sequence. It is known [2] that if $r \neq 1$, the $T(r)$ matrix has the inverse $T(p)$ where $p = r/(r-1)$. We therefore have

$$t_n = \sum_{m=n}^{\infty} \binom{m}{n} r_n^{m-n}(1 - r_n)^{n+1} \sum_{k=m}^{\infty} (1 - p)^{m+1} \binom{k}{m} p^{k-m} u_k.$$

If we formally interchange the order of summation we have

$$t_n = \sum_{k=1}^{\infty} (1 - r_n)^{n+1} u_k \sum_{m=n}^{k} \binom{m}{n}\binom{k}{m} r_n^{m-n} p^{k-m} (1 - p)^{m+1}.$$

The interchange of summation will be justified provided the series

$$R = \sum_{k=n}^{\infty} | 1 - r_n |^{n+1} | u_k | \sum_{m=n}^{k} \binom{m}{n}\binom{k}{m} | r_n |^{m-n} | p |^{k-m} | 1 - p |^{m+1}$$

converges. Since the sequence $\{u_k\}$ converges there exists a constant M such that $|u_k| \leqq M$ $(k=0, 1, \cdots)$. Therefore

$$R \leqq M \sum_{k=n}^{\infty} | 1 - r_n |^{n+1} \binom{k}{n} \sum_{m=n}^{k} \binom{k-n}{m-n} | r_n |^{m-n} | p |^{k-m} | 1 - p |^{m+1}$$

$$\leqq M \sum_{k=n}^{\infty} | 1 - r_n |^{n+1} \binom{k}{n} | 1 - p |^{n+1} [| r_n | \, | 1 - p | + | p |]^{k-n}$$

$$\leqq M \frac{| 1 - r_n |^{n+1} | 1 - p |^{n+1}}{[1 - (| r_n | |1 - p| + | p |)]^{n+1}},$$

provided $|r_n| |1-p| + |p| < 1$. That is, provided

$$(3.1) \qquad | r_n | + | r | < | 1 - r | .$$

Consequently, if (3.1) holds we have

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (1 - r_n)^{n+1} \; u_k \sum_{v=0}^{k-n} \binom{k-n}{v} r_n^v p^{k-n-v} (1 - p)^{n+v+1}$$

$$= \sum_{k=n}^{\infty} \binom{k}{n} (1 - r_n)^{n+1}(1 - p)^{n+1}(r_n - r_n p + p)^{k-n} u_k$$

$$= \sum_{k=n}^{\infty} \binom{k}{n} [1 - (r_n + p - r_n p)]^{n+1}(r_n - r_n p + p)^{k-n} u_k$$

$$= \sum_{k=n}^{\infty} \binom{k}{n} (1 - x_n)^{n+1} x_n^{k-n} u_k,$$

where $x_n = r_n + p - r_n p = (r - r_n)/(r - 1)$. It is clear that $\{t_n\}$ will converge to s, and hence $T(r_n) \supset T(r)$, if the matrix $T(x_n)$ is regular. This will be true if the conditions of Theorem 2.1 are satisfied for $T(x_n)$. We have therefore the result:

THEOREM 3.1. *If* $r \neq 1$ *is a complex constant such that* (3.1) *holds then* $T(r_n) \supset T(r)$ *if*

$$\left| \frac{r_n - r}{1 - r} \right| < 1 \qquad (n = 0, 1, \cdots), \tag{3.2}$$

and

$$(n + 1) \log \frac{|1 - r_n|}{|1 - r| - |r_n - r|} < M \qquad (n = 0, 1, \cdots), \tag{3.3}$$

where M *is a constant independent of* n.

We may note that if $r_n = r_2 = \text{constant}$ $(n = 0, 1, \cdots)$, then (3.2) and (3.3) hold if and only if

$$|1 - r_2| = |1 - r| - |r_2 - r|. \tag{3.4}$$

This gives:

COROLLARY 3.2. *If* $r \neq 1$, $|r_2| + |r| < |1 - r|$, *and if* (3.4) *holds, then* $T(r_2) \supset T(r)$.

This last result was first proved by Laush [3], who also proved that the conditions of Corollary 3.2 are necessary if $0 < |r| < |r_2| < 1$.

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LEHIGH UNIVERSITY