AN EXTENSION OF THE TAYLOR SUMMABILITY TRANSFORM

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1. Introduction. Two matrix methods of summability which have been the object of much recent research are the Euler matrix, \( E(r) = (a_{nk}) \), and the Taylor matrix, \( T(r) = (b_{nk}) \). The elements of these matrices are defined by

\[
a_{nk} = \begin{cases} \binom{n}{k} r^k (1 - r)^{n-k} & (k \leq n), \\ 0 & (k > n), \end{cases}
\]

and

\[
b_{nk} = \begin{cases} 0 & (k < n), \\ \frac{k}{n} r^n (1 - r)^{n+1} & (k \geq n), \end{cases}
\]

respectively.

R. P. Agnew [1] has studied the matrix \( E(r_n) \) which is obtained from the \( E(r) \) matrix by replacing the parameter \( r \) by a parameter \( r_n \) which is dependent on \( n \). This matrix has also been studied by Wuyts-Torfs [4], [5]. It is natural to form the matrix \( T(r_n) \) from the \( T(r) \) matrix in a similar manner. The elements \( c_{nk} \) of the \( T(r_n) \) matrix are then defined by

\[
c_{nk} = \begin{cases} 0 & (k < n), \\ \frac{k}{n} r_n^k (1 - r_n)^{n+1} & (k \geq n). \end{cases}
\]

In §2 we determine necessary and sufficient conditions on \( r_n \) in order that \( T(r_n) \) be regular. In §3 we determine sufficient conditions on \( r_n \) which insure that \( T(r_n) \supseteq T(r) \), i.e., which insure that each sequence which is summable \( T(r) \) is summable \( T(r_n) \) to the same value.

2. Regularity of \( T(r_n) \). A matrix \( A = (a_{nk}) \) is regular if and only if the well-known Silverman-Toeplitz conditions:

\[
(2.1) \quad \lim_{n} a_{nk} = 0 \quad (k = 0, 1, 2, \ldots),
\]

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1 The results of this note appear in the author’s doctoral dissertation written at the University of Kentucky in 1962 under the direction of Professor V. F. Cowling.
Condition (2.1) holds trivially for the $T(r_n)$ matrix without restriction on $r_n$. Condition (2.2) is satisfied if and only if $|r_n| < 1$ \((n = 0, 1, \cdots)\) since
\[
\sum_{n=0}^{\infty} \binom{k}{n} r_n^{k-n} (1 - r_n)^{n+1} = 1 \quad (n = 0, 1, \cdots),
\]
when $|r_n| < 1$ and the series diverges if $|r_n| \geq 1$. Furthermore, if $|r_n| < 1$,
\[
\sum_{n=0}^{\infty} \binom{k}{n} |r_n|^{k-n} |1 - r_n|^{n+1} = \left[ \frac{1 - r_n}{1 - |r_n|} \right]^{n+1},
\]
so that (2.3) will hold if and only if
\[
\left[ \frac{1 - r_n}{1 - |r_n|} \right]^{n+1} \leq K \quad (n = 0, 1, \cdots),
\]
where $K$ is a constant independent of $n$. That is, (2.3) will hold if and only if $\exp[(n+1) \log(|1 - r_n|/(1 - |r_n|))]$ is a bounded function of $n$. We therefore have the result:

**Theorem 2.1.** The $T(r_n)$ matrix is regular if and only if $|r_n| < 1$ \((n = 0, 1, \cdots)\) and \((n+1) \log(|1 - r_n|/(1 - |r_n|))\) is a bounded function of $n$.

The above theorem may be given the following geometrical characterization:

**Theorem 2.2.** Let $M$ be a non-negative constant and let $k_n = \exp(M/(n+1))$ \((n = 0, 1, \cdots)\). Let $I_n$ denote the inner loop of the limaçon $L_n$ given by:
\[
\rho_n = \frac{k_n^2 \cos \phi_n - k_n}{k_n^2 - 1},
\]
where $(\rho_n, \phi_n)$ are polar coordinates at $z = 1$ with $\phi_n$ measured clockwise from the line segment joining $z = 0$ and $z = 1$. Then $T(r_n)$ is regular if and only if $|r_n| < 1$ and $r_n \in I_n$ \((n = 0, 1, \cdots)\).
Proof. Condition (2.4) holds if and only if
\[ \left| \frac{1 - r_n}{1 - \left| r_n \right|} \right| \leq e^{M(n+1)} \quad (n = 0, 1, \cdots), \]
where $M$ is a non-negative constant. For fixed $n$ this last inequality is satisfied in a region bounded by the curve $L_n$ given by
\[ |1 - r_n| = k_n(1 - |r_n|). \]

Introducing polar coordinates $(\rho_n, \phi_n)$ as described in the statement of Theorem 2.2, we have
\[ 1 - r_n = \rho_n(\cos \phi_n + i \sin \phi_n). \]
Equation (2.6) becomes
\[ \rho_n = k_n[1 - |1 - \rho_n(\cos \phi_n + i \sin \phi_n)|]. \]
Hence
\[ (\rho_n - k_n)^2 = k_n^2(1 + 2k_n \cos \phi_n + \rho_n^2), \]
so that
\[ \rho_n(k_n^2 - 1) - 2k_n^2 \cos \phi_n + 2k_n = 0, \]
which reduces to (2.5).

For fixed $n$ this is the equation of a limaçon with two branches through the point $z=1$. Since $|r_n| < 1$, in order that $T(r_n)$ be regular, it is clear that $r_n$ must lie in the inner loop $I_n$ of $L_n$. This proves the theorem.

It is useful to note that the sequence of regions $I_n$ approaches the interval $[0, 1]$ as $n \to \infty$. To see this we notice that $\rho_n = 2k_n/(k_n + 1)$ when $\phi_n = 0$, and that this quantity tends to 1 as $n \to \infty$. It is an exercise in elementary calculus to prove that the maximum height of the boundary of $I_n$ above the x-axis tends to 0 as $n \to \infty$.

We may note also that, in particular, $T(r_n)$ is regular if $0 \leq r_n < 1$ ($n = 0, 1, \cdots$). Furthermore, if $r_n = r$, a constant ($n = 0, 1, \cdots$), then $T(r_n) = T(r)$ is regular if and only if $0 \leq r < 1$. This is the known necessary and sufficient condition that $T(r)$ be regular [2].

3. The relation $T(r_n) \supset T(r)$. Let $\{s_i\}$ be a sequence which is $T(r)$-summable to $s$, where $r \neq 1$. Let $\{u_k\}$ denote the $T(r)$-transform of the sequence $\{s_i\}$ and let $\{t_k\}$ denote the $T(r_n)$-transform of the same sequence. It is known [2] that if $r \neq 1$, the $T(r)$ matrix has the inverse $T(\rho)$ where $\rho = r/(r - 1)$. We therefore have
\[ t_n = \sum_{m=n}^{\infty} \binom{m}{n} r_n^{m-n} (1 - r_n)^{n+1} \sum_{k=m}^{\infty} (1 - p)^{m+1} \binom{k}{m} p^{k-m} u_k. \]

If we formally interchange the order of summation we have

\[ t_n = \sum_{k=1}^{\infty} (1 - r_n)^{n+1} u_k \sum_{m=n}^{\infty} \binom{m}{n} \binom{k}{m} r_n^{m-n} p^{k-m} (1 - p)^{m+1}. \]

The interchange of summation will be justified provided the series

\[ R = \sum_{k=1}^{\infty} (1 - r_n)^{n+1} |u_k| \sum_{m=n}^{\infty} \binom{m}{n} \binom{k}{m} |r_n|^{m-n} |p|^{k-m} |1 - p|^{m+1} \]

converges. Since the sequence \(|u_k|\) converges there exists a constant \(M\) such that \(|u_k| \leq M\) for \(k = 0, 1, \ldots\). Therefore

\[ R \leq M \sum_{k=1}^{\infty} (1 - r_n)^{n+1} \binom{k}{n} \sum_{m=n}^{\infty} \binom{k}{m} |r_n|^{m-n} |p|^{k-m} |1 - p|^{m+1} \]

\[ \leq M \sum_{k=1}^{\infty} (1 - r_n)^{n+1} \binom{k}{n} |1 - p|^{n+1} \left[ |r_n| |1 - p| + |p| \right]^{k-n} \]

provided \(|r_n| |1 - p| + |p| < 1\). That is, provided

\[ (3.1) \quad |r_n| + |r| < |1 - r|. \]

Consequently, if (3.1) holds we have

\[ t_n = \sum_{k=1}^{\infty} \binom{k}{n} (1 - r_n)^{n+1} u_k \sum_{v=0}^{k-n} \binom{k-n}{v} r_n v^{k-n-v} (1 - p)^{n+1} \]

\[ = \sum_{k=1}^{\infty} \binom{k}{n} (1 - r_n)^{n+1} (1 - p)^{n+1} (r_n - r_n p + p)^{k-n} u_k \]

\[ = \sum_{k=1}^{\infty} \binom{k}{n} [1 - (r_n + p - r_n p)]^{n+1} (r_n - r_n p + p)^{k-n} u_k \]

\[ = \sum_{k=1}^{\infty} \binom{k}{n} (1 - x_n)^{n+1} x_n^{k-n} u_k, \]

where \(x_n = r_n + p - r_n p = (r - r_n)/(r - 1)\). It is clear that \(\{t_n\}\) will converge to \(s\), and hence \(T(r_n) \supseteq T(r_n)\), if the matrix \(T(x_n)\) is regular. This will be true if the conditions of Theorem 2.1 are satisfied for \(T(x_n)\). We have therefore the result:
Theorem 3.1. If $r \neq 1$ is a complex constant such that (3.1) holds then $T(r_n) \supset T(r)$ if

$$\frac{|r_n - r|}{1 - r} < 1 \quad (n = 0, 1, \ldots),$$

and

$$\frac{1 - r_n}{1 - r} - \frac{|r_n - r|}{1 - |r_n - r|} < M \quad (n = 0, 1, \ldots),$$

where $M$ is a constant independent of $n$.

We may note that if $r_n = r_2 = \text{constant} (n = 0, 1, \ldots)$, then (3.2) and (3.3) hold if and only if

$$|1 - r_2| = |1 - r| - |r_2 - r|.$$  

This gives:

**Corollary 3.2.** If $r \neq 1$, $|r_2| + |r| < |1 - r|$, and if (3.4) holds, then $T(r_n) \supset T(r)$.

This last result was first proved by Laush [3], who also proved that the conditions of Corollary 3.2 are necessary if $0 < |r| < |r_2| < 1$.

**References**

5. ———, *On a generalization of the Euler limit method*, Simon Stevin 33 (1959), 27–33. (Dutch)

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