ON COMMUTATORS OF OPERATORS ON HILBERT SPACE

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1. In this note we first generalize a result of P. R. Halmos [3] concerning commutators of (bounded) operators on Hilbert space. Then we obtain some partial results on a problem of commutators in von Neumann algebras which is closely related to another problem raised by Halmos in [4]. Let \( \mathcal{H} \) be any (infinite-dimensional) Hilbert space, and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded operators on \( \mathcal{H} \). We follow Halmos [3] in calling a subspace \( \mathcal{H} \subset \mathcal{H} \) large if \( \mathcal{H} \) contains infinitely many orthogonal copies of \( \mathcal{H} \). Halmos proved in [3] that any operator in \( \mathcal{L}(\mathcal{H}) \) with a large reducing null space is a commutator (of two bounded operators in \( \mathcal{L}(\mathcal{H}) \)). We generalize this to

**Theorem 1.** Any operator in \( \mathcal{L}(\mathcal{H}) \) which has a large null space is a commutator.

The construction involved in the proof of this theorem is a generalization of Halmos' construction in [3], and our construction actually yields a slightly more general result than Theorem 1. This more general result does not admit a nice formulation on nonseparable spaces, but on separable spaces it is easy to describe.

**Theorem 2.** Let \( \mathcal{H} \) be a separable Hilbert space and let \( \mathcal{H} = \mathcal{H} \oplus \mathcal{H} \). If this decomposition of \( \mathcal{H} \) is used to write every operator \( T \in \mathcal{L}(\mathcal{H}) \) as a 2×2 operator matrix

\[
T = \begin{pmatrix} A & C \\ B & D \end{pmatrix},
\]

where the entries are operators on \( \mathcal{H} \), then every operator \( T \in \mathcal{L}(\mathcal{H}) \) of the form

\[
T = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix},
\]

where \( C \) is a compact operator, is a commutator.

For separable spaces, Theorem 1 is a special case \( (C=0) \) of Theorem 2, and the proof of Theorem 1 for nonseparable spaces is an easy modification of the proof of that theorem for separable spaces. Thus we confine ourselves to proving Theorem 2.

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Proof of Theorem 2. Let \( \{E_i\}_{i=1}^{\infty} \) be any countable collection of mutually orthogonal projections \( E_i \in \mathcal{L}(\mathcal{K}) \) such that the sum of the \( E_i \) is the identity operator on \( \mathcal{K} \) and such that the range of each \( E_i \) is an infinite-dimensional subspace of \( \mathcal{K} \). Each \( E_i \) gives rise to a projection \( F_i \in \mathcal{L}(\mathcal{K}) \) defined by

\[
F_i = \begin{pmatrix} 0 & 0 \\ 0 & E_i \end{pmatrix}, \quad i = 1, 2, \ldots,
\]

and if \( F_0 \in \mathcal{L}(\mathcal{K}) \) is defined as

\[
F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

then the \( F_i, i = 0, 1, 2, \ldots, \) are mutually orthogonal projections on \( \mathcal{K} \) whose sum is the identity operator on \( \mathcal{K} \). Furthermore, the \( F_i \) are mutually equivalent in the sense of Murray-von Neumann in the von Neumann (v.N.) algebra \( \mathcal{L}(\mathcal{K}) \). Thus the \( F_i \) together with the implementing partial isometries form a complete set of matrix units for \( \mathcal{L}(\mathcal{K}) \), and we can use this set of matrix units to regard \( \mathcal{L}(\mathcal{K}) \) as an infinite matrix algebra. More precisely, it follows from [2, Proposition 5, p. 27], that \( \mathcal{L}(\mathcal{K}) \) is unitarily equivalent to the v.N. algebra \( \mathcal{A} \) of all \( \mathbb{N}_0 \times \mathbb{N}_0 \) matrices with entries from the v.N. algebra \( \mathcal{L}_0 \mathcal{L}(\mathcal{K}) \mathcal{L}_0 \cong \mathcal{L}(\mathcal{K}) \) which act as operators on the Hilbert space \( \mathcal{K}_1 = \mathcal{K} \oplus \mathcal{K} \oplus \cdots \). (Recall that \( \mathcal{L}_0(\mathcal{K}) = \mathbb{K} \oplus 0 \).) Thus we can and do work with the infinite matrices of \( \mathcal{A} \) instead of the \( 2 \times 2 \) matrices of \( \mathcal{L}(\mathcal{K}) \). It is easy to see that any operator in \( \mathcal{L}(\mathcal{K}) \) of the form

\[
\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}
\]

is carried by the isomorphism between \( \mathcal{L}(\mathcal{K}) \) and \( \mathcal{A} \) onto an operator in \( \mathcal{A} \) of the form

\[
X = \begin{pmatrix} A & C_1 & C_2 & C_3 & \cdots \\ B_1 & 0 & 0 & 0 & \cdots \\ B_2 & 0 & 0 & \cdots & \cdots \\ B_3 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.
\]

A matrix calculation shows that formally \( X \) is the commutator

\[
X = SR - RS
\]

where
It is obvious that $R$ represents a bounded operator in $A$, and thus to complete the proof it suffices to show that if $C$ is compact then the collection $\{E_i\}_{i=1}^n$ can be chosen in such a way as to ensure that $S$ represents a bounded operator. Since $X$ is a bounded operator and since, obviously, the matrix

$$
\begin{pmatrix}
0 & A & 0 & \\
0 & A & 0 & \\
0 & A & 0 & \\
\end{pmatrix}
$$

represents a bounded operator, it is easy to see that it suffices to show that the collection $\{E_i\}$ can be chosen so that the "Toeplitz" matrix

$$
Y = \begin{pmatrix}
C_1 & C_2 & C_3 & \\
0 & C_1 & C_2 & \\
\vdots & 0 & C_1 & \\
\vdots & \vdots & \vdots & \\
\end{pmatrix}
$$

represents a bounded operator. That this can be done follows from the following sequence of lemmas.
LEMMA 1.1. If $C$ is a compact operator on the separable Hilbert space $\mathcal{K}$, and $C = UP$ is the polar decomposition of $C$, then there exists a sequence of mutually orthogonal infinite-dimensional subspaces $\mathcal{K}_1, \mathcal{K}_2, \ldots \subset \mathcal{K}$ such that

(a) $\sum_{i=1}^{\infty} \mathcal{K}_i = \mathcal{K}$;
(b) each $\mathcal{K}_i$ is a reducing subspace for $P$, so that the linear manifolds $\{C(\mathcal{K}_i)\}$ are mutually orthogonal;
(c) for each $i \geq 2$, $\|P \mid \mathcal{K}_i\| = \|C \mid \mathcal{K}_i\| < 1/2^i$.

Proof. If $P$ has an infinite-dimensional null space $\mathcal{N}$, take $\mathcal{K}_1$ to be the direct sum of $\mathcal{N} \oplus \mathcal{N}$ and a sufficiently large subspace $\mathcal{M} \subset \mathcal{K}$ to ensure that $\mathcal{K}_1$ is infinite-dimensional. Arrange it so that $\mathcal{N} \oplus \mathcal{M}$ remains infinite-dimensional, and write $\mathcal{N} \oplus \mathcal{M} = \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \cdots$, where $\mathcal{K}_2, \mathcal{K}_3, \cdots$ are also infinite-dimensional. If $\mathcal{N}$ is finite-dimensional, then

$$P = \sum_{i=1}^{\infty} \alpha_i E_i,$$

where each $\alpha_i$ is a positive scalar, $\{\alpha_i\} \to 0$, and the $E_i$ are mutually orthogonal projections on finite-dimensional spaces. Now the problem is essentially that of partitioning a countable set into a countable union of disjoint infinite subsets, maintaining some care so that (c) will be satisfied. We omit further details of that argument.

LEMMA 1.2. With $C, \mathcal{K}$ and the sequence $\{\mathcal{K}_i\}$ as in Lemma 1.1, for each positive integer $i$, let $E_i \in \mathcal{L}(\mathcal{K})$ be the projection on $\mathcal{K}_i$. If the collection $\{E_i\}_{i=1}^{\infty}$ is used to determine a unitary equivalence between $\mathcal{L}(\mathcal{K})$ and $A$ as above, then the operators $C_i \in \mathcal{L}(\mathcal{K})$ which appear in the matrix $X$ have mutually orthogonal ranges and in addition satisfy

$$\sum \|C_i\|^2 < \infty.$$

Proof. By definition [2, Proposition 5, p. 27], $C_i$ is the restriction to $\mathcal{K} \oplus 0 \subset \mathcal{K}$ of an operator $F_iTU_i \in \mathcal{L}(\mathcal{K})$, where $T$ is as in (1) and $U_i \in \mathcal{L}(\mathcal{K})$ is a partial isometry with initial space $\mathcal{K} \oplus 0 \subset \mathcal{K}$ and final space $0 \oplus \mathcal{K}_i \subset \mathcal{K}$. Each $U_i$ obviously has a $2 \times 2$ matrix

$$U_i = \begin{pmatrix} 0 & 0 \\ V_i & 0 \end{pmatrix},$$

where $V_i \in \mathcal{L}(\mathcal{K})$ satisfies $V_iV_i^* = E_i$ and $V_i^*V_i = 1_{\mathcal{K}}$.

By multiplying the appropriate $2 \times 2$ matrices we obtain

$$F_iTU_i = \begin{pmatrix} CV_i & 0 \\ 0 & 0 \end{pmatrix}.$$
Thus for \( x, y \in \mathcal{H} \oplus 0 \) and \( i \neq j \), \((C_i x, C_j y) = (F_i T U_i x, F_j T U_j y) = (C V x, C V y) = 0 \) since \( V x \in \mathcal{H}_i \) and \( V y \in \mathcal{H}_j \). This proves that \( C_i \) and \( C_j \) have orthogonal ranges. Also, for \( i \geq 2 \) and \( \|x\| = 1 \), \( \|C_i x\|^2 = (C V x, C V x) = \|C V x\|^2 \leq \|C \| \|x\|^2 \leq 1/2^i \), so that clearly
\[
\sum_i \|C_i\|^2 < \infty.
\]

To complete the proof of Theorem 2, it now suffices to show that if the operators \( C_i \in \mathcal{L}(\mathcal{H}) \) which appear in the matrix \( Y \) have mutually orthogonal ranges and satisfy
\[
\sum_i \|C_i\|^2 < \infty,
\]
then \( Y \) is bounded. An easy computation which we omit shows that indeed this is the case, and in fact
\[
\|Y\| \leq \sum_i \|C_i\|^2.
\]

As an immediate corollary of Theorem 1 we obtain

**Corollary 1.3 (Halmos).** Any operator \( A \) on a Hilbert space \( \mathcal{H} \) is the sum of two commutators.

**Proof.** Write \( \mathcal{H} = \mathcal{H} \oplus \mathcal{H} \) and write \( A \) as the sum of two operators each of which vanishes on one copy of \( \mathcal{H} \).

**Conjecture.** The author conjectures that Theorem 2 remains true if the restriction of compactness is removed from the operator \( C \).

2. In [4], Halmos raised the question of whether every operator on a separable Hilbert space which is not a scalar modulo the compact operators is a commutator. If the answer to this question is yes, and we denote the closed ideal of compact operators by \( \mathcal{C} \), then the \( C^* \)-algebra \( \mathcal{L}(\mathcal{H})/\mathcal{C} \) has the following property:

(S) Every nonscalar element of the algebra is a commutator of two elements in the algebra.

Calkin [1] imbeds \( \mathcal{L}(\mathcal{H})/\mathcal{C} \) in a v.N. algebra \( \mathcal{C} \) (acting on a non-separable Hilbert space), and thus a related question is whether \( \mathcal{C} \), or for that matter any v.N. algebra, has property (S). A partial answer is given by the following theorem.

**Theorem 3.** Every v.N. algebra \( \mathcal{A} \) which is not a factor of type III contains a projection \( E \neq 1 \) which is not a commutator in \( \mathcal{A} \); thus, no such \( \mathcal{A} \) has property (S).

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1 See Remark (3) at the end of the paper.
Proof. If \( A \) is not a factor, it follows from Kleinecke's result in [6] that every nonzero central projection fails to be a commutator in \( A \), so that it suffices to consider the case that \( A \) is a factor. If \( A \) is a finite factor, then \( A \) possesses a numerical-valued trace, and thus any projection with nonzero trace fails to be a commutator in \( A \). If \( A \) is a factor of type \( I_m \), then \( A \) is algebraically isomorphic to the algebra of all bounded operators on some Hilbert space \( \mathcal{H} \) and thus \( A \) contains the proper closed ideal \( \mathcal{S} \) of compact operators on \( \mathcal{H} \). It follows that any projection of the form \( 1 - E \) where \( E \) is a finite-dimensional projection cannot be a commutator in \( A \). (If \( 1 - E \) were a commutator in \( A \), then the identity element \( 1 + \mathcal{S} \) of the Banach algebra \( A/\mathcal{S} \) would be a commutator in \( A/\mathcal{S} \), which is impossible [4].) Finally, if \( A \) is a factor of type \( II_{\infty} \), let \( \mathcal{F} \) be the subset of \( A \) consisting of all elements which are of "finite rank" in the sense of [7, Definition 1.2.1, p. 97]. It follows from [7, Lemma 1.2.1, p. 97] that \( \mathcal{F} \) is a two-sided ideal in \( A \), and thus the uniform closure \( \mathcal{F}_0 \) of \( \mathcal{F} \) is a proper closed two-sided ideal in \( A \). If \( E \) is any finite projection in \( A \), then \( E \in \mathcal{F} \) and again \( 1 - E \) cannot be a commutator in \( A \), which completes the argument.

Whether there is a type III factor with property (S) or not, the author does not know. However, the following theorem indicates that perhaps every type III factor has property (S).

Theorem 4. If \( A \) is a factor of type III, then every \( A \in \mathcal{A} \) which has a nontrivial null space is a commutator in \( A \). In particular, every projection \( P \neq 1 \) in \( A \) is a commutator in \( A \).

Proof. One knows that the projection \( E \) on the null space of \( A \) is an element of \( \mathcal{A} \). It follows from repeated application of Lemma 4.12 of [5] and the fact that all nonzero projections in \( \mathcal{A} \) are equivalent that there exists a countable family \( \{F_i\} \) of mutually orthogonal, equivalent projections in \( \mathcal{A} \) such that

\[
\sum_i F_i = E.
\]

If we adjoin \( 1 - E \) to the family \( \{F_i\} \) we obtain a countable family of mutually orthogonal, equivalent projections in \( \mathcal{A} \) whose sum is 1. An application of [2, Proposition 5, p. 27] yields a unitary isomorphism of \( \mathcal{A} \) onto the v.N. algebra \( \mathcal{B} \) of all \( \mathbb{K}_0 \times \mathbb{K}_0 \) operator matrices with entries from the algebra \( (1 - E)\mathcal{A}(1 - E) \), and under this isomorphism...
A is carried onto a matrix of the form of $X$ in Theorem 2, where $C_1 = C_2 = \cdots = 0$. But then $X = SR - RS$, just as in Theorem 2.

3. Remarks. (1) I wish to express my appreciation to Professor Paul Halmos for stimulating my interest in commutators and to Don Deckard for many interesting conversations on the subject.

(2) Calkin conjectured in [1] that the v.N. algebra which he constructed to contain $\mathcal{L}(\mathcal{C})/\mathcal{C}$ is a factor of type III. Theorems 3 and 4 lend support to that conjecture.

(3) Added in proof. Since this paper was written, Arlen Brown and the author have completely settled the question of which operators on Hilbert space are commutators. Theorem 1 of the present paper proved useful in that connection. (See: Structure theorem for commutators of operators, Arlen Brown and Carl Pearcy, Bull. Amer. Math. Soc. 70 (1964), 779–780.) We have also proved that when $\mathcal{C}$ is separable, every factor of type III on $\mathcal{C}$ has property (S).

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