ON CROSSINGS OF ARBITRARY CURVES BY CERTAIN GAUSSIAN PROCESSES[1]

M. R. LEADBETTER

1. **Introduction.** Much has been written concerning the mean number of crossings of a fixed level u, by a stationary Gaussian stochastic process, in a given time T. Most recently (and most rigorously), this problem has been considered by Bulinskaya [2], who derives the well-known formula

$$E[N(T)] = \frac{T}{\pi}\left(\frac{-r''(0)}{r(0)}\right)^{1/2} e^{-u^2/[2r(0)]} \tag{1}$$

under conditions which are very close to the necessary ones. Here $N(T)$ is the number of crossings of the level u in $(0, T)$ by the stationary Gaussian process $x(t)$, with covariance function $r(\tau)$. The symbol E denotes expectation. The treatment of this problem given by Bulinskaya is essentially a rigorization of the method used by Grenander and Rosenblatt [4], which in turn extends an argument due to Kac [7].

For some applications, however, it is important to consider crossings of a curve, rather than a fixed level, by such a process, and to consider also the same problem for certain Gaussian, but nonstationary processes. In §2 we shall obtain the formula corresponding to (1) for the case where $\{x(t)\}$ is a stationary Gaussian process and $u = u(t)$ is an arbitrary (differentiable) curve, instead of a fixed level. In §4, the same problem will be considered for a nonstationary process $z(t) = \int_0^t x(s)ds$ where $\{x(t)\}$ is, as before, a stationary Gaussian process. The discussion of this $z(t)$-process has application to the study of the probabilistic behaviour of certain physical systems. It would be possible to state a result corresponding to (1) for curve crossings by a member of a wide class of (nonstationary) Gaussian processes. However, this could be stated in very general terms only, and moreover it is obvious from the derivation for the $z(t)$-case how such a general result would be formulated.

2. **Curve-crossings by a stationary Gaussian process.** Throughout this section we consider a separable, stationary Gaussian process,

Received by the editors August 12, 1963.

[1] Research supported by National Aeronautics and Space Administration Contract NASw-334 with the Research Triangle Institute.

having zero mean, covariance function $r(\tau)=E\{x(t)x(t+\tau)\}$ and (integrated) spectrum $F(\lambda)$. That is

$$r(\tau) = \int_0^\infty \cos \tau\lambda \, dF(\lambda).$$

Define

$$\sigma_0^2 = \int_0^\infty dF(\lambda) = r(0),$$

$$\sigma_2^2 = \int_0^\infty \lambda^2 \, dF(\lambda) = -r''(0).$$

We shall assume that the spectrum $F(\lambda)$ does not have its entire increase at $\lambda=0$ (hence $\sigma_2^2>0$), and furthermore satisfies the condition

$$\int_0^\infty \lambda^2[\log(1+\lambda)]^b \, dF(\lambda) < \infty \quad \text{for some} \quad b > 1. \tag{2}$$

It then follows from Hunt [6] that, with probability one, the sample function of the process has a derivative $x'(t)$, which is everywhere continuous. A statement of Hunt's result, in a convenient form for our purposes, has been given by Belaev [1].

We shall assume also that the curve $u(t)$ has a continuous first derivative $u'(t)$ in $0 \leqq t \leqq T$. Let a new process $\{x_*(t): 0 \leqq t \leqq T\}$ be defined by $x_*(t)=x(t)-u(t)$. Then $x_*(t)$ is a normal process whose sample functions have continuous first derivatives with probability one, and crossings of the curve $u(t)$ by the process $x(t)$ are equivalent to "axis crossings" by the process $x_*(t)$. Let $N_x(T)$ denote the number of crossings of the curve $u(t)$ by $x(t)$, or equivalently, the number of axis crossings by $x_*(t)$, in $0<t<T$. Since the one-dimensional distributions of $x_*(t)$ have bounded densities (being univariate normal densities), it follows from Bulinskaya [2, Theorem 1], that $N_x(T)$ is finite with probability one and that the probability of $x_*(t)$ touching the axis in $0 \leqq t \leqq T$ is zero.

In evaluating the mean of $N_x(T)$, there is no loss of generality in taking $T=1$. We shall write N_x for $N_x(1)$ and proceed by a series of lemmas to obtain its mean. It is convenient to follow the method of Bulinskaya [2] in defining a process consisting of straight line segments, and approximating (in our case) the $x_*(t)$-process as follows:

For a given integer n let $\alpha_k=k/2^n$, $k=0, 1, \cdots, 2^n$. Define a process $\{y_n(t)\}$ by

$$(3)\quad y_n(t) = x_*(\alpha_k) + 2^n[x_*(\alpha_{k+1}) - x_*(\alpha_k)](t - \alpha_k), \text{ when } \alpha_k \leqq t \leqq \alpha_{k+1}$$
$$(4)\qquad = A_k + B_k t, \text{ say.}$$

That is, the $y_n(t)$-process consist of a series of straight lines with vertices at points 2^{-n} apart.

LEMMA 1. *Write N_{y_n} for the number of axis crossings by $y_n(t)$ in $0<t<1$. Then $E\{N_{y_n}\}\to E\{N_x\}$ as $n\to\infty$.*

PROOF. It is clear that $N_{y_n}\leqq N_x$ and that $N_{y_n}\to N_x$ with probability one. The theorem concerning the convergence of integrals of increasing sequences of non-negative functions (cf. Halmos [5, p. 112]), then yields the result.

LEMMA 2. *Let $\{\delta_v(y)\}$ be a sequence of non-negative integrable functions such that $\int_{-\infty}^{\infty}\delta_v(x)\,dx=1$ and $\int_{-\lambda}^{\lambda}\delta_v(x)\,dx\to 1$ as $v\to\infty$ for any fixed $\lambda>0$. (We may call $\{\delta_v\}$ a "δ-function sequence".) Then, with probability one,*

$$N_{y_n} = \lim_{v\to\infty}\int_0^1 \delta_v\{y_n(t)\}\,|\,y_n'(t)\,|\,dt$$

and $\int_0^1\delta_v\{y_n(t)\}\,|y_n'(t)|\,dt\leqq 2^n$.

PROOF. With $\alpha_k=k/2^n$ as defined previously we have from (4),

$$(5)\qquad \begin{aligned}\int_0^1 \delta_v\{y_n(t)\}\,|\,y_n'(t)\,|\,dt &= \sum_{k=0}^{2^n-1}\int_{\alpha_k}^{\alpha_{k+1}}\delta_v(A_k+B_k t)\,|\,B_k\,|\,dt\\ &= \sum_{k=0}^{2^n-1}\left|\int_{y_n(\alpha_k)}^{y_n(\alpha_{k+1})}\delta_v(x)\,dx\right|.\end{aligned}$$

With probability one $y_n(\alpha_k)$ is not zero for any $k=0, 1, \cdots, 2^n$. From the assumed δ-function properties it follows that if $y_n(\alpha_k)$ and $y_n(\alpha_{k+1})$ have the same sign the corresponding integral tends to zero. Otherwise this integral converges to ± 1. Thus if the interval (α_k, α_{k+1}) contains a zero of $y_n(t)$, the corresponding term in the sum tends to one, and otherwise it tends to zero. Hence the first part of the lemma follows. The second part follows from (5) since each term in the sum is dominated by $\int_{-\infty}^{\infty}\delta_v(x)dx=1$.

In virtue of Lemmas 1 and 2, it follows by dominated convergence, and using Fubini's theorem for positive functions, that

$$E\{N_{y_n}\} = \lim_{v\to\infty} \int_0^1 E\{\delta_v(y_n(t)) \mid y_n'(t) \mid \} dt \tag{6}$$
$$= \lim_{v\to\infty} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mid w \mid \delta_v(v) f_n(v, w) dv dw dt$$

where

$$f_n(v, w) = (2\pi D^{1/2})^{-1}$$
$$\cdot \exp\left[\frac{-1}{2D}\{C(x+\alpha)^2 - 2B(x+\alpha)(y+\beta) + A(y+\beta)^2\}\right], \tag{7}$$
$$\alpha = -E\{y_n(t)\}, \quad \beta = -E\{y_n'(t)\}, \qquad A = \text{var}\{y_n(t)\},$$
$$B = \text{var}\{y_n'(t)\}, \quad C = \text{cov}\{y_n(t), y_n'(t)\}, \quad D = AC - B^2.$$

(These quantities also depend on n and t. This dependence will be exhibited when necessary by writing $\alpha_n(t)$ for α, etc.)

Up to this point we have used δ_v-functions with general properties only, and this practice could be continued. However, it is more convenient to use now a particular δ-function sequence. If $h(y)$ is a nonnegative L_1 function whose integral is unity, the sequence defined by $\delta_v(t) = vh(vt)$ is a δ-function sequence. If we take $h(y) = 1$ for $|y| < 1$ and $h(y) = 0$ otherwise we get, essentially, the sequence used by Kac [7]. We shall here use a "normal" form, viz., $h(y) = (2\pi)^{-1/2}e^{-y^2/2}$. (This δ-function has been used previously on similar problems, for example by Steinberg et al. [8].)

Using this form for $\delta_v(y)$, (6) becomes

$$E\{N_{y_n}\} = \lim_{v\to\infty} (2\pi)^{-1/2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-v^2/2} \mid w \mid f_n(v/v, w) dv dw dt. \tag{8}$$

The integrand in (8) is dominated by

$$(\mid w \mid / D^{1/2}) e^{-v^2/2} \exp\{(-1/2C)(y+\beta)^2\},$$

and converges to $|w| e^{-v^2/2} f_n(0, w)$ as $v \to \infty$. Hence, by dominated convergence we have

LEMMA 3. *$E\{N_{y_n}\} = \int_0^1 \int_{-\infty}^{\infty} |w| f_n(0, w) dw dt$, where $f_n(0, w)$ is obtained from equation* (7).

In order to use Lemmas 1 and 3 we must investigate the limiting behaviour of $f_n(0, w)$.

LEMMA 4. *We have the following uniform limits in $0 \leq t \leq 1$:*

(i) $A_n(t) \to r(0)$, (ii) $B_n(t) \to 0$,

(iii) $C_n(t) \to -r''(0)$, (iv) $D_n(t) \to -r(0)r''(0) > 0$,

(v) $\alpha_n(t) \to u(t)$, (vi) $\beta_n(t) \to u'(t)$.

PROOF. Write $k_n = k_n(t)$ for the unique integer k such that $k/2^n \leqq t < (k+1)/2^n$ $(0 \leqq t \leqq 1)$. From (3) we have

$$\alpha_n(t) = u(k_n/2^n) + 2^n\{u[(k_n+1)/2^n] - u(k_n/2^n)\}(t - k_n/2^n),$$
$$\beta_n(t) = 2^n\{u[(k_n+1)/2^n] - u(k_n/2^n)\}.$$

But

$$u[(k_n+1)/2^n] = u[k_n/2^n] + 2^{-n}u'(k_n/2^n + \lambda_n), \qquad 0 < \lambda_n < 2^{-n}.$$

Hence,

$$\begin{aligned} 2^n\{u[(k_n+1)/2^n] - u(k_n/2^n)\} \\ = u'(k_n/2^n + \lambda_n) \to u'(t) \text{ uniformly in } 0 \leqq t \leqq 1, \end{aligned} \tag{9}$$

since $k_n/2^n \to t$, $\lambda_n \to 0$ both uniformly in t and u' is a uniformly continuous function of t in $0 \leqq t \leqq 1$. Statements (v) and (vi) follow at once from (9).

The variances and covariance have the same form as those obtained by Bulinskaya, viz.,

$$\begin{aligned} A_n(t) &= r(0)[(1 - 2^nt + k_n)^2 + (2^nt - k_n)^2] \\ &\quad + 2r(2^{-n})(2^nt - k_n)(1 - 2^nt + k_n), \\ B_n(t) &= 2^n[(2^nt - k_n) - (1 - 2^nt + k_n)][r(0) - r(2^{-n})], \\ C_n(t) &= 2^{2n+1}[r(0) - r(2^{-n})]. \end{aligned}$$

Writing $r(2^{-n}) = r(0) + 2^{-(2n+1)}r''(\theta_n)$ $(0 < \theta_n < 2^{-n},\ r'(0) = 0)$ and noting that $t - k_n/2^n \leqq 2^{-n}$, $(k_n+1)/2^n - t \leqq 2^{-n}$ it is easy to show that

$$\begin{aligned} |A_n(t) - r(0)| &\leqq 2^{-2n}|r''(\theta_n)|, \\ |B_n(t)| &\leqq 2^{-n}|r''(\theta_n)|, \\ |C_n(t) + r''(0)| &= |r''(\theta_n) - r''(0)|, \end{aligned}$$

from which the uniform limits (i)–(iv) (and hence the truth of the lemma) follow.

From Lemma 3, and equation (7), we obtain

$$\begin{aligned} E\{N_{v_n}\} = (2\pi D^{1/2})^{-1} \int_0^1 e^{-\alpha^2/(2A)}dt \int_0^\infty v \Big\{ &\exp\left[-\frac{A}{2D}(v + \gamma(D/A)^{1/2})^2\right] \\ &+ \exp\left[-\frac{A}{2D}(v - \gamma(D/A)^{1/2})^2\right]\Big\} dv \end{aligned}$$

where $\gamma = \gamma_n(t) = (A/D)^{1/2}(\beta - B\alpha/A)$, and a little reduction yields

$$(10)\quad E\{N_{v_n}\} = A^{-1}(2D/\pi)^{1/2}\int_0^1 e^{-\alpha^2/(2A)}\{\phi(\gamma) + \gamma[\Phi(\gamma) - 1/2\}]dt$$

where $\phi(y)$ is the normal density function $(2\pi)^{-1/2}e^{-y^2/2}$ and $\Phi(y) = \int_{-\infty}^{\infty}\phi(t)dt$. Now from Lemma 4 it follows that

$$\gamma = \gamma_n(t) \to u'(t)/\{-r''(0)\}^{1/2}$$

and since the convergence is uniform, $\gamma_n(t)$ is uniformly bounded in $0 \leq t \leq 1$. Hence by bounded convergence, using the limits of Lemma 4,

$$E\{N_{v_n}\} \to [-r''(0)/r(0)]^{1/2}$$

$$(11)\quad \int_0^1 \phi[u(t)/(r(0))^{1/2}]\{2\phi[-u'(t)/(-r''(0))^{1/2}] - [u'(t)/(-r''(0))^{1/2}][2\Phi\{-u'(t)/(-r''(0))^{1/2}\} - 1]\}dt,$$

as $n \to \infty$.

Finally we may now state the following result.

THEOREM 1. *Let $x(t)$ be a separable, stationary Gaussian process whose spectrum $F(\lambda)$ satisfies (2) and does not have its entire increase at $\lambda = 0$. Let $u(t)$ possess a continuous derivative $u'(t)$ in $0 \leq t \leq T$. Then with the notation already defined*

$$(12)\quad E\{N_x(T)\} = \sigma_0^{-1}\int_0^T \phi(u(t)/\sigma_0)[2\sigma_2\phi(u'(t)/\sigma_2) + u'(t)\{2\Phi(u'(t)/\sigma_2) - 1\}]dt.$$

This result follows from (11), using Lemma 1.

3. **Examples.** (i) In the case where $u(t) = u$, a constant, the second integral in (12) vanishes, and the first reduces to give the standard result (1).

(ii) For a "straight line barrier," with gradient $b \neq 0$, $u(t) = a + bt$,

$$EN_x(T) = \sigma_0^{-1}[2\sigma_2\phi(b/\sigma_2) + b(2\Phi(b/\sigma_2) - 1)]\int_0^T \phi[(a+bt)/\sigma_0]dt$$

$$= 2\frac{\sigma_2}{b}[\phi(b/\sigma_2) + 2\Phi(b/\sigma_2) - 1][\Phi((a+bT)/\sigma_0) - \Phi(a/\sigma_0)]$$

4. **Curve-crossings by the integral of a stationary Gaussian process.** Suppose now that $x(t)$ is, as before, a separable stationary Gaus-

sian process with zero mean, covariance function $r(\tau)$, and spectrum $F(\lambda)$, where $F(\lambda)$ is now assumed to satisfy

$$\int_0^\infty [\log(1+\lambda)]^b dF(\lambda) < \infty \quad \text{for some} \quad b > 1, \tag{13}$$

instead of (2). This condition guarantees continuity of the sample function $x(t)$, with probability one. For convenience, we shall also suppose that $\text{var}\{x(t)\}=1$. Define $z(t)=\int_0^t x(s)ds$. Then under the conditions assumed, the following formulae are valid.

$$\begin{aligned} E\{z(t)\} &= 0, \\ E\{z^2(t)\} &= \sigma^2(t) = \int_0^t \int_0^t r(u-v)\, du dv \\ &= 2\int_0^\infty \frac{1-\cos t\lambda}{\lambda^2}\, dF(\lambda), \\ E\{x(t)z(t)\} &= \sigma(t)\Psi(t) = \int_0^t r(u)\, du \\ &= \int_0^\infty \frac{\sin t\lambda}{\lambda}\, dF(\lambda). \end{aligned} \tag{14}$$

Equations (14) have been given by Cramér [3], who has also shown that if $N_z(T)$ denotes the number of crossings of the fixed level C by $z(t)$ in $0 \leq t \leq T$, then

$$E\{N_z(T)\} = \int_0^T Q(t)\, dt, \tag{15}$$

where

$$\begin{aligned} Q(t) = \frac{1}{2\pi\sigma} e^{-C^2/(2\sigma^2)} \Bigg[& 2(1-\Psi^2)^{1/2} \exp\left\{\frac{-C^2\Psi^2}{2\sigma^2(1-\Psi^2)}\right\} \\ & + (2\pi)^{1/2}\left[1 - 2\Phi\left(\frac{-C\Psi}{\sigma(1-\Psi^2)^{1/2}}\right)\right]\Bigg]. \end{aligned} \tag{16}$$

For the case of curve-crossings by the process $z(t)$ we give the following result.

THEOREM 2. *Let $u(t)$ have a continuous derivative $u'(t)$ in $0 \leq t \leq T$. With the notation above, the mean number of crossings of $u(t)$ by $z(t)$ in $(0, T)$ is given by* (15) *where now $Q(t)$ has the form, writing u, σ, Ψ, for $u(t)$, $\sigma(t)$, $\Psi(t)$, respectively,*

$$(17) \qquad Q(t) = \sigma^{-1}(1 - \Psi^2)^{1/2}\phi(u/\sigma)[2\phi(\eta) + \eta(2\Phi(\eta) - 1)]$$

in which

$$\eta = \eta(t) = (u'(t) - u\Psi/\sigma)/(1 - \Psi^2)^{1/2}.$$

PROOF. The proof follows the same pattern as that in §2. In fact if we let $\{y_n(t)\}$ be the process consisting of line segments, as before, but derived from $z(t) - u(t)$, we obtain equations (6), (7) for $E\{N_{y_n}\}$. The quantities α, β have the same form as before and therefore have again the uniform limits $u(t)$, $u'(t)$, respectively, as $n \to \infty$.

On the other hand we have, from the definition of the $y_n(t)$-process,

$$A_n(t) = \sigma^2(k_n/2^n)(1 - 2^nt + k_n)^2 + \sigma^2[(k_n + 1)/2^n](2^nt - k_n)^2 + (k_n + 1 - 2^nt)(2^nt - k_n)[\sigma^2(k_n/2^n) + \sigma^2[(k_n + 1)/2^n] - \sigma^2(2^{-n})]$$

where use has been made of the fact that

$$\operatorname{cov}(z(t), z(u)) = \tfrac{1}{2}[\sigma^2(t) + \sigma^2(u) - \sigma^2(t - u)],$$

a formula given by Cramér [3], and easily derivable from the definition of the $z(t)$-process. Hence

$$(18) \qquad \begin{aligned} A_n(t) = {} & \sigma^2(k_n/2^n)(k_n + 1 - 2^nt) + \sigma^2[(k_n + 1)/2^n](2^nt - k_n) \\ & - \sigma^2(2^{-n})(k_n + 1 - 2^nt)(2^nt - k_n). \end{aligned}$$

Now from (14),

$$\sigma^2(t) = t^2 \int_0^\infty \frac{1 - \cos t\lambda}{(t\lambda)^2/2}\, dF(\lambda)$$

and it is thus clear from dominated convergence that

$$(19) \qquad \sigma^2(t) \sim t^2 \quad \text{as} \quad t \to 0.$$

The third term on the right of (18) is dominated by $\sigma^2(2^{-n})$ and thus tends to zero (uniformly in t) as $n \to \infty$. On the other hand since $\sigma^2(t)$ has (from (14)) the continuous derivative $2\sigma(t)\Psi(t)$, we may write

$$(20) \qquad \sigma^2(k_n/2^n) = \sigma^2(t) - 2(t - k_n/2^n)\sigma(t_1)\Psi(t_1), \qquad |t_1 - t| < 2^{-n},$$

with a similar expression for $\sigma^2[(k_n+1)/2^n]$. It follows that the first two terms on the right of (18) may be written as $\sigma^2(t) + E_n(t)$, where for some constant K, $|E_n(t)| \leqq 2^{-n}K$, in $0 \leqq t \leqq 1$. Hence it follows that $A_n(t) \to \sigma^2(t)$, uniformly in $0 \leqq t \leqq 1$, as $n \to \infty$.

Correspondingly, we have, for B,

$$(21) \qquad \begin{aligned} B_n(t) = {} & 2^{n-1}[\sigma^2[(k_n + 1)/2^n] - \sigma^2(k_n/2^n)] \\ & - 2^{n-1}\sigma^2(2^{-n})[1 - 2(2^nt - k_n)]. \end{aligned}$$

From (20) and the (uniform) continuity of $\sigma(t)\Psi(t)$, it follows that the first term in (21) converges uniformly to $\sigma(t)\Psi(t)$ as $n\to\infty$, whereas (by use of (19)) the second term tends uniformly to zero. Hence $B_n(t)\to\sigma(t)\Psi(t)$, uniformly in $0\leqq t\leqq 1$.

Finally for C we have

$$C_n(t) = 2^{2n}\sigma^2(2^{-n}) \to 1 \text{ as } n\to\infty, \text{ by (19).}$$

With these uniform limits for A, B, C, α, β (and hence the limit $\sigma^2(t)(1-\Psi^2(t))$ for D), we may proceed exactly as in §2. Hence the truth of the theorem follows.

5. **Acknowledgement.** The author expresses his sincere gratitude to Professor Harald Cramér for valuable conversations concerning this and a variety of related problems.

References

1. Yu. K. Belaev, *Continuity and Hölder's conditions for sample functions of stationary Gaussian processes*, Proc. 4th Berkeley Sympos., Vol. 2, pp. 23–33, Univ. of California Press, Berkeley, Calif., 1961.

2. E. C. Bulinskaya, *On the mean number of crossings of a level by a stationary Gaussian process*, Teor. Verojatnost. i Primenen. **6** (1961), 474–477. (Russian)

3. Harald Cramér, *On extreme values of certain normal nonstationary stochastic processes*, Research Triangle Inst. Tech. Rep. GU-68, No. 4, 1963.

4. U. Grenander and M. Rosenblatt, *Statistical analysis of stationary time series*, Wiley, New York, 1957, p. 272.

5. P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, N. J., 1950.

6. G. A. Hunt, *Random Fourier transforms*, Trans. Amer. Math. Soc. **71** (1951), 38–69.

7. M. Kac, *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **49** (1943), 314–320.

8. H. Steinberg, P. M. Schultheiss, C. A. Wogrin, and F. Zweig, *Short time frequency measurement of narrow-band random signals by means of a zero counting process*, J. Appl. Phys. **26** (1955), 195–201.

Research Triangle Institute, Durham, North Carolina