$a^t = az$, a contradiction. Hence aG^{p^n}, bG^{p^n} are not conjugate, and the theorem is proved.

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SOME REMARKS ON THE DIOPHANTINE EQUATION $x^3+y^3+z^3=x+y+z$

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In order to avoid certain trivial solutions of the Diophantine equation $x^3+y^3+z^3=x+y+z$ we initially assume $x \geqq y \geqq 0$, $z<0$ and $x \neq -z$. All letters will indicate rational integers throughout, with the exception of k, t, M which denote rational numbers. S. L. Segal [1] has shown that if $x=y$ then only finitely many solutions are forthcoming. Generalizing the method of Segal slightly we prove the following result:

THEOREM 1. *If A, B, C and D are given nonzero integers satisfying $(A, B)=(C, D)=1$ and $C \mid A^2$ then the Diophantine equation*

$$A(Cx^3 - Dx) = B(z^3 - z) \tag{1}$$

has just a finite number of solutions (x, z). An upper bound for the number of solutions is given by

$$5 \sum_{c \mid B(A^2D^3 - CB^2)A^3C^3} (2\sigma(c) + 1), \tag{2}$$

where $\sigma(n)$ denotes the sum of the positive divisors of the natural number n.

PROOF. If $(x, z)=d$ then we write $x=ad$, $z=bd$ with $(a, b)=1$. Upon substituting for x and z in the given equation and dividing by d we obtain the equation

$$(ACa^3 - Bb^3)d^2 = ADa - Bb \tag{3}$$

from which it follows that $ADa-Bb=cd^2$ for some suitable c. Upon writing

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$$ACa^3d^2 - ADa = Bb^3d^2 - Bb = \frac{(Bb)^3d^2}{B^2} - Bb,$$

substituting $Bb = ADa - cd^2$ and simplifying, we obtain

$$(4) \qquad A(A^2D^3 - CB^2)a^3 = 3A^2D^2a^2cd^2 - 3ADac^2d^4 + c^3d^6 - B^2c.$$

Further,

$$ACx^3 - Bz^3 = ADx - Bz = d(ADa - Bb) = cd^3 = (ACa^3 - Bb^3)d^3$$

from which it follows that

$$(5) \qquad ACa^3 - Bb^3 = c.$$

Upon writing $A^2 = CE$ for a suitable E, multiplying (5) by $(ED^3 - B^2)$ and making use of (4) we obtain

$$(6) \quad B(ED^3 - B^2)b^3 = c(3A^2D^2a^2d^2 - 3ADacd^4 + c^2d^6 - ED^3)$$

so that $c \mid B(ED^3 - B^2)b^3$. If $(b, c) = F$ so that $c = FG$ and $b = FH$ with $(G, H) = 1$ then from (5) it follows that $F \mid AC$. The relation $c \mid B(ED^3 - B^2)b^3$ may be rewritten as $FG \mid B(ED^3 - B^2)F^3H^3$ from which we obtain $G \mid B(ED^3 - B^2)F^2$ since $(G, H) = 1$. Thus $c = FG \mid B(ED^3 - B^2)F^3 \mid B(A^2D^3 - CB^2)A^3C^3$ since $F \mid AC$.

If a value of c, say c_0, is specified and a particular solution (a_0, b_0) of $ACa^3 - Bb^3 = c_0$ is found then the original equation becomes

$$A(Ca_0^3d^3 - Da_0d) = B(b_0^3d^3 - b_0d)$$

from which it follows that there can arise at most one value of d for any particular specified triple (a_0, b_0, c_0) having the above-mentioned properties. It remains to obtain an upper bound for the number of solutions of an equation of type (5) in which c has been specified. An important result of B. Delauney [2] and T. Nagell [3] states that the Diophantine equation

$$(7) \qquad pX^3 + qX^2Y + rXY^2 + sY^3 = 1$$

has at most five solutions, provided that the discriminant

$$(8) \qquad \Delta = q^2r^2 + 18pqrs - 4pr^3 - 4sq^3 - 27p^2s^2$$

of the binary cubic form is negative. From this result it can further be shown [4] that $ACa^3 - Bb^3 = c_0$ has at most $5|c_0|$ solutions with the property $(a, c_0) = 1$. The discriminant of equation (5) is $-27A^2B^2C^2 < 0$ so that the above-mentioned results apply. To get an estimate for the number of solutions of (5) for which $(a, c_0) > 1$ we let $(a, c_0) = I$. Then $a = mI$ and $c_0 = nI$ with $(m, n) = 1$. It follows from equation (5) together with the fact that $(a, b)=1$ that $I|B$. If $B=sI$ then we obtain the equation

$$ACI^2m^3 - sb^3 = n \tag{9}$$

in which $(m, n)=(m, b)=1$. Thus equation (9) has at most $5|n|$ solutions. The cases $(a, c_0)=1$ and $(a, c_0)>1$ can be disposed of together if we allow n to run through all divisors of c_0. Hence the total number of solutions of equation (5) for a fixed value of c, say c_0, does not exceed $5(2\sigma(c_0)+1)$, $\sigma(w)$ denoting the sum of the positive divisors of the natural number w. Thus the total number of solutions (x, z) of the original equation does not exceed

$$5\sum(2\sigma(c) + 1)\ c\,|\ B(A^2D^3 - CB^2)A^3C^3.$$

Reverting now to the original equation, S. D. Chowla et al. [5] has shown that there exist infinitely many solutions of $x^3+y^3+z^3=x+y+z$ which satisfy the further conditions

$$x + y + z = m, \tag{10}$$

$$x + y = 3m, \tag{11}$$

and

$$x + z \neq 0, \tag{12}$$

where m is a natural number which adopts all values satisfying $m^2\equiv 1 \pmod 9$.

Using Chowla's method we exhibit two further infinite classes of nontrivial solutions of the original equation which satisfy the conditions (10) and (12) together with the condition

$$x + y = km. \tag{13}$$

The manner in which we choose the values for k and m is now described.

From the equation $x^3+y^3+z^3=x+y+z$ we obtain the equations

$$\begin{aligned} x^3 + y^3 &= (x + y)(x^2 - xy + y^2) = km[(x + y)^2 - 3xy] \\ &= km(k^2m^2 - 3xy) = m - (1 - k)^3m^3 \end{aligned} \tag{14}$$

from which it follows that

$$xy = \frac{m^2 - 1}{3k} + m^2(k - 1) \tag{15}$$

and therefore also

$$(x-y)^2=(k-2)^2m^2-\frac{4(m^2-1)}{3k}\cdot \tag{16}$$

For integer solutions we require from (13) that km be an integer. Upon assuming further that $3k\mid(m^2-1)$ it follows that (15) and (16) represent integers. It is now assumed further that $3k=t^2$ for some positive rational number t. From equation (16) we obtain

$$(tw)^2-Mm^2=4, \tag{17}$$

where $w=x-y$ may be taken to be positive without loss of generality and where $M=3k(k-2)^2-4$. Upon writing $k=a/b$ with $(a, b)=1$ we rewrite (17) in the form

$$(3ab)^2w^2-Qm^2=12ab^3, \tag{18}$$

where $Q=3a[4(a-b)^3-a^3]$. In order that the above Pellian equation have any solutions (w, m) it is necessary that Q be positive which, in turn, requires $k>8/3$.

In Chowla's case $k=3$ so that (18) becomes

$$(3w)^2-5m^2=4. \tag{19}$$

Every solution of (19) necessarily has $m^2\equiv 1 \pmod 9$ so that xy and $(x-y)^2$ are integral in every case. In any solution of (19) in which $m\equiv 0 \pmod 2$ we also have $w\equiv 0 \pmod 2$. Thus in any such case it follows that $x+y=3m$ and $x-y=w$ are both even integers so that x and y are both integral as required. In the same way, any solution of (19) in which $m\equiv 1 \pmod 2$ must have $w\equiv 1 \pmod 2$ which implies that $x+y$ and $x-y$ are both odd integers. Hence for all such solutions x and y are integral.

The equation $X^2-5Y^2=1$ has $(9, 4)$ as its fundamental solution. The general solution is then given by $X+Y\sqrt{5}=\pm(9+4\sqrt{5})^n$, n integral. Hence if $n\equiv 1 \pmod 2$ it follows that $X\equiv 0 \pmod 3$. Since the double of any solution of $X^2-5Y^2=1$ is a solution of $X^2-5Y^2=4$ it follows that there are infinitely many solutions of (19).

In order to obtain solutions of $x^3+y^3+z^3=x+y+z$ with $z<0$ it is necessary to choose $m>0$. Upon squaring (11) and adding it to (16) in the case $k=3$ we obtain

$$2(x^2+y^2)=9m^2+\frac{5m^2}{9}+\frac{4}{9}\cdot \tag{20}$$

Since $y=3m-x$ we obtain a quadratic equation in x from which it follows that $2x\sim(3+\sqrt{5}/3)m$. In the same way we obtain $2y\sim(3-\sqrt{5}/3)m$. Thus we have shown that $x^3+y^3+z^3=x+y+z$ has infinitely many solutions (x, y, z) satisfying $x>y>0$, $z<0$ and the conditions imposed by equations (10) through (12).

If $k=12$ then clearly $x+y\equiv 0 \pmod 2$ holds, $t=6$, $M=3596$ and $m^2\equiv 1 \pmod{36}$ is required. Equation (17) can be simplified and becomes

$$(3w)^2 - 899m^2 = 1. \tag{21}$$

Since $x+y\equiv 0 \pmod 2$, it is necessary that $w\equiv 0 \pmod 2$ if x and y are both to be integral. Any solution (w, m) of (21) in which $w\equiv 0 \pmod 2$ necessarily has $m^2\equiv 1 \pmod{36}$, as required by equation (15), since $-899\equiv 1 \pmod{36}$. Since the fundamental solution of $X^2-899Y^2=1$ is $(30,1)$ we obtain infinitely many solutions of this equation with $X\equiv 0 \pmod 6$ by taking $X+Y\sqrt{899}=\pm(30+\sqrt{899})^n$ with $n\equiv 1 \pmod 2$. Hence equation (21) has infinitely many solutions (w, m) in which $w\equiv 0 \pmod 2$ as required. As in the treatment of $k=3$ here we can show that $2x\sim(12+\sqrt{111})m$, $2y\sim(12-\sqrt{111})m$ and of course $z=-11m$. By choosing $n=3$ we obtain the solution $(39579, 3609, -39589)$.

The second infinite class of nontrivial solutions has been found by choosing $k=16/3$. In this case we necessarily have $m\equiv 0 \pmod 3$. Equations (15) and (16) require, respectively, that m have the properties $m^2\equiv 33 \pmod{48}$ and $m^2\equiv 9 \pmod{36}$. Thus if equations (15) and (16) are both to be satisfied it is necessary to have $m^2\equiv 81 \pmod{144}$. This last condition makes it certain that the requirement $m\equiv 0 \pmod 3$ will always be satisfied. The values $t=4$ and $9M=1564$, when substituted into equation (17), lead to the equation

$$(6w)^2 - 391m^2 = 9. \tag{22}$$

From equation (13), using the facts that $k=16/3$ and $m\equiv 0 \pmod 3$, it follows that $x+y\equiv 0 \pmod 2$ and so $w\equiv 0 \pmod 2$ is necessary. If (w, m) is any solution of equation (22) in which $w\equiv 0 \pmod 2$ it follows that we must have $m^2\equiv 81 \pmod{144}$, as previously required.

The Pellian equation $X^2-391Y^2=1$ is known to have $(7338680, 371133)$ as its fundamental solution [6]. Using this fact it can be shown, by using previously employed methods, that the equation $X^2-391Y^2=1$ has infinitely many solutions (X, Y) in which $X\equiv 0 \pmod 4$. It follows that equation (22) has infinitely many solutions (w, m) in which $w\equiv 0 \pmod 2$. Finally we can show that

$$2x\sim\frac{m}{3}\left(16+\frac{\sqrt{391}}{2}\right),\qquad 2y\sim\frac{m}{3}\left(16-\frac{\sqrt{391}}{2}\right)$$

and of course

$$z = \frac{-13m}{3} \cdot$$

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