Here we have used (2.5).

I wish to thank the referee for some improvements in §2 of this paper.

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MULTIPLE TRANSITIVITY OF PRIMITIVE PERMUTATION GROUPS

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Introduction. We wish to consider two theorems on permutation groups which are a generalization of a two-part theorem found in [1, pp. 66–67], a theorem concerned with a permutation group on a finite set. We shall remove the restriction of finiteness.

The following notation and definitions will be used in the discussion:

If Y is a set, then $|Y|$ denotes the cardinal number of Y. If g is a mapping of X into Z and $x \in X$, then xg denotes the image of x under g. If $Y \subseteq X$, then $(Y)g$ denotes the image of Y under g. If G is a group and $H \leqq G$ and $g \in G$, then $H^g = g^{-1}Hg$. A permutation group G on a set X is said to be r-ply (or r-fold) transitive on $Y \subseteq X$ (where r is a nonzero cardinal number and $|Y| \geqq r$), if (i) for every pair of subsets A and B of Y of cardinal number r and for any one-to-one map f of A onto B, there exists $g \in G$ such that $g|_A = f$ (where $g|_A$ is the restriction of g to A), and if (ii) $(Y)g = Y$ for every $g \in G$. A 1-fold transitive group on $Y \subseteq X$ is called transitive on Y.

A permutation group $G \neq 1$ on a set X is said to be imprimitive on $Y \subseteq X$ if: (i) Y can be written as a disjoint union of two or more nonvoid sets $\{S_\alpha\}_{\alpha \in \mathfrak{A}}$, (ii) for at least one $\alpha \in \mathfrak{A}$, $|S_\alpha| \geqq 2$ and (iii) given $g \in G$ and $\alpha \in \mathfrak{A}$, then $(S_\alpha)g = S_\beta$ for some $\beta \in \mathfrak{A}$. The sets S_α, $\alpha \in \mathfrak{A}$, are called imprimitivity sets of G on Y. If G is not imprimitive on Y, then G is called primitive on Y.

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The theorem which we shall generalize is:

Let G be a permutation group on n letters which is primitive, and let H be a transitive subgroup of G on m letters (where $m \geqq 2$) fixing the remaining $n-m$ letters. Then (1) if H is primitive, G is $(n-m+1)$-fold transitive; (2) in any event, G is doubly transitive.

We shall prove:

THEOREM 1. *Let G be a primitive permutation group on X where $|X| \geqq 3$. Suppose that there exists a subgroup $H \leqq G$ which* (i) *is primitive and r-fold transitive on a subset $Y \subseteq X$ where $|Y| \geqq 2$ and where $N = |X \backslash Y|$ is a positive integer, and which* (ii) *fixes the elements of $X \backslash Y$ (i.e., if $x \in X \backslash Y$, $g \in H$, then $xg = x$). Then G is $(N+r)$-fold transitive on X.*

We shall not consider the case where N is an infinite cardinal number.

THEOREM 2. *A permutation group G on a set X where $|X| \geqq 3$ is doubly transitive on X if and only if* (i) *G is primitive,* (ii) *there exists a subgroup $H \leqq G$ which is transitive on some $Y \subseteq X$, where $|Y| \geqq 2$ and $X \backslash Y$ is finite and nonvoid, and* (iii) *H fixes the elements of $X \backslash Y$.*

It should be noted that: (1) if G is m-ply transitive on X, where $m = r+N$, $r \geqq 1$ and $N \geqq 1$, then there exists a subgroup $H \leqq G$, which is r-fold transitive on $Y \subseteq X$, where $|X \backslash Y| = N$ and H fixes the elements of $X \backslash Y$; (2) if G is m-fold transitive on X where $m \geqq 2$, then G is primitive [1, pp. 56 and 66]. Hence we have the following partial converse to Theorem 1: If G is an $(N+r)$-fold transitive permutation group on X, where $N \geqq 1$, $r \geqq 2$, then G is primitive, and there is a subgroup $H \leqq G$, which (i) is primitive and r-fold transitive on some $Y \subseteq X$, where $|Y| \geqq 2$ and $N = |X \backslash Y|$, and which (ii) fixes the elements of $X \backslash Y$. It is also true that a nontrivial primitive permutation group G on a set X is transitive on X [1, p. 64].

We shall need the following propositions:

PROPOSITION 1. *Let G be a permutation group on X and let $H \leqq G$.* (1) *If H is r-fold transitive on $Y \subseteq X$ and fixes the elements of $X \backslash Y$, then H^g (where $g \in G$) is r-fold transitive on $(Y)g$ and fixes $(X \backslash Y)g = X \backslash (Y)g$.* (2) *If H is primitive on $Y \subseteq X$, then H^g is primitive on $(Y)g$ for every $g \in G$.*

PROPOSITION 2. *Let G be a permutation group on X, and let H_1, $H_2 \leqq G$ such that* (i) *H_j is transitive on $Y_j \subseteq X$ and fixes $X \backslash Y_j$, $j = 1, 2$, and* (ii) *$Y_1 \cap Y_2 \neq \varnothing$. Then $H = \{H_1, H_2\}$ is transitive on $Y = Y_1 \cup Y_2$ and fixes $X \backslash Y$.*

PROPOSITION 3. *Let G be an s-fold transitive permutation group on X, and suppose that there is a subgroup $H \leqq G$ such that* (i) *H is r-fold transitive on $Y \subseteq X$,* (ii) *H fixes $X \setminus Y$ and* (iii) *$s = |X \setminus Y|$. Then G* is *$(r+s)$-fold transitive on X.*

The proofs of these propositions are simple [1, pp. 56, 66–67].

PROPOSITION 4. *Let $Y \subseteq X$ such that $X \setminus Y$ is finite, and let f be a permutation of X. Let $C = Y \cap (Y)f$, $A = Y \setminus C$ and $B = (Y)f \setminus C$. Then* (i) *A, B and C are pairwise disjoint,* (ii) *A and B are finite with $|A| = |B| \leqq |X \setminus Y|$ and* (iii) *$Y = C \cup A$, $(Y)f = C \cup B$.*

PROOF. Since $(X \setminus Y)f = X \setminus (Y)f$, $|X \setminus Y| = |X \setminus (Y)f|$. Let $W = Y \cup (Y)f$. Then $X \setminus Y = (X \setminus W) \cup ((Y)f \setminus C)$ and $X \setminus (Y)f = (X \setminus W) \cup (Y \setminus C)$ where $(X \setminus W) \cap ((Y)f \setminus C) = \varnothing$ and $(X \setminus W) \cap (Y \setminus C) = \varnothing$. Hence $|X \setminus Y| = |X \setminus W| + |(Y)f \setminus C|$ and $|X \setminus (Y)f| = |X \setminus W| + |Y \setminus C|$. From the finiteness of the above numbers, we infer $|A| = |B| \leqq |X \setminus Y|$.

PROOF OF THEOREM 1. We use induction on N. If $N = 1$, the desired result follows for all cardinal numbers $r \geqq 1$ by Proposition 3. Suppose that the result is true for $N_0 \geqq 1$ and for all $r \geqq 1$, and assume that $N = N_0 + 1$. We wish to show the existence of an $H_0 \leqq G$ such that H_0 is $(r+1)$-fold transitive on some $Y_0 \subseteq X$ where $|Y_0| \geqq 2$, $N_0 = |X \setminus Y_0|$ and H_0 fixes $X \setminus Y_0$ (then H_0 will be at least doubly transitive and hence primitive); and thus, by the induction hypothesis, G is $N_0 + (r+1) = (N+r)$-fold transitive on X.

Consider all the sets $(Y)g$, $g \in G$. If for every $g_1, g_2 \in G$, $(Y)g_1 \cap (Y)g_2 = \varnothing$ or $(Y)g_1 = (Y)g_2$, then the family of sets $\{(Y)g: g \in G\}$ (by the transitivity of G on X and the fact $Y \neq X$ and $|Y| \geqq 2$) would constitute imprimitivity sets of G on X, a contradiction. Hence there exist $g_1, g_2 \in G$ such that $(Y)g_1 \cap (Y)g_2 \neq \varnothing$ and $(Y)g_1 \neq (Y)g_2$. Thus $Y \cap (Y)(g_2 g_1^{-1}) \neq \varnothing$ and $Y \neq (Y)(g_2 g_1^{-1})$, giving a $g_3 \in G$ such that $Y \cap (Y)g_3 \neq \varnothing$ and $Y \neq (Y)g_3$. Let $C_3 = Y \cap (Y)g_3$, $A_3 = Y \setminus C_3$ and $B_3 = (Y)g_3 \setminus C_3$. One of the sets A_3, B_3 is nonvoid. Let $s_3 = |A_3|$. By Proposition 4, we have (i) $Y = C_3 \cup A_3$, $(Y)g_3 = C_3 \cup B_3$, (ii) A_3, B_3 and C_3 are pairwise disjoint and (iii) $s_3 = |B_3|$ and s_3 is finite. Hence A_3, B_3 and C_3 are all nonvoid, $|X \setminus [Y \cup (Y)g_3]| = |(X \setminus Y) \setminus B_3| = |X \setminus Y| - |B_3| = N - s_3$, and $s_3 \geqq 1$.

Choose $g_0 \in G$ such that $Y \cap (Y)g_0 \neq \varnothing$, $Y \neq (Y)g_0$ and $s_0 = |Y \setminus [Y \cap (Y)g_0]|$ is minimal ($s_0 \geqq 1$). Let $C_0 = Y \cap (Y)g_0$, $A_0 = Y \setminus C_0$, and $B_0 = (Y)g_0 \setminus C_0$. By Proposition 2, $H_0 = \{H, H^{g_0}\}$ is transitive on $Y_0 = Y \cup (Y)g_0 = Y \cup B_0$ and fixes $X \setminus Y_0$. Suppose that $s_0 = 1$. Since $H \leqq H_0$ is r-fold transitive on $Y \subseteq Y_0$ and fixes $Y_0 \setminus Y = B_0$, then (by Proposition 3 with $s = s_0 = 1$) H_0 is $(r+1)$-fold transitive on

Y_0 and fixes $X\setminus Y_0=(X\setminus Y)\setminus B_0$, where $|X\setminus Y_0|=N-s_0=N_0$. Furthermore, since $Y\subseteq Y_0$, $|Y_0|\geqq 2$. That is, if $s_0=1$ then G is $(N+r)$-fold transitive on X.

In fact, $s_0=1$. For, suppose, that $s_0>1$. $(Y)g_0=C_0\cup B_0$, where $B_0=\{b_1, \cdots, b_{s_0}\}$ and $C_0\neq\emptyset$. Since H^{g_0} is primitive on $(Y)g_0$ and $|B_0|=s_0\geqq 2$, there is an element $h^1\in H^{g_0}$ which maps one or more (but not all) of the elements of B_0 onto elements of B_0 and the remaining elements of B_0 onto elements of C_0: i.e., by renumbering if necessary, $b_1, \cdots, b_u$ (where $1\leqq u<s_0$) are mapped onto elements of B_0, and $b_{u+1}, \cdots, b_{s_0}$ are mapped onto elements of C_0 by h^1. Since h^1 maps $(Y)g_0$ onto itself and B_0 is finite, precisely (s_0-u) elements of B_0 are the images of (s_0-u) elements of C_0, say $c_1, \cdots, c_{s_0-u}$. Hence the elements of $C_0\setminus\{c_1, \cdots, c_{s_0-u}\}$ are mapped by h^1 onto $C_0\setminus(\{b_{u+1}, \cdots, b_{s_0}\})h^1\subseteq C_0$. Since h^1 fixes A_0 $(A_0\subseteq X\setminus(Y)g_0)$, we have

$$\begin{aligned}(Y)h^1 &= (C_0\cup A_0)h^1 = A_0\cup(C_0)h^1\\ &= A_0\cup(C_0\setminus\{c_1, \cdots, c_{s_0-u}\})h^1\cup(\{c_1, \cdots, c_{s_0-u}\})h^1\\ &= A_0\cup[C_0\setminus(\{b_{u+1}, \cdots, b_{s_0}\})h^1]\cup(\{c_1, \cdots, c_{s_0-u}\})h^1.\end{aligned}$$

But since $Y\cap(\{c_1, \cdots, c_{s_0-u}\})h^1=\emptyset$, then

$$Y\cap(Y)h^1=(A_0\cup C_0)\setminus(\{b_{u+1}, \cdots, b_{s_0}\})h^1\neq\emptyset.$$

But $(Y)h^1\neq Y$ and $Y\setminus[Y\cap(Y)h^1]=(\{b_{u+1}, \cdots, b_{s_0}\})h^1$ which has cardinal number $s_0-u<s_0$. This contradicts the minimality of s_0, and, hence, $s_0=1$.

PROOF OF THEOREM 2. We wish to show that there is a subgroup $K_0\leqq G$ which has the properties: (i) K_0 is transitive on some subset Y_{K_0} of X, (ii) K_0 fixes $X\setminus Y_{K_0}$ and (iii) $|X\setminus Y_{K_0}|=1$. Then Theorem 2 will follow from Proposition 3.

Let $\mathcal{K}$ be the collection of all subgroups $K\leqq G$ such that: (i) K is transitive on some subset $Y_K\subseteq X$, (ii) K fixes the elements of $X\setminus Y_K$, (iii) $|Y_K|\geqq 2$ and (iv) $X\setminus Y_K$ is a finite, nonvoid set. Clearly $H\in\mathcal{K}$. Furthermore there is a subgroup $K_0\in\mathcal{K}$ such that $|X\setminus Y_{K_0}|\leqq|X\setminus Y_K|$ for every $K\in\mathcal{K}$. To complete the proof we now show that $|X\setminus Y_{K_0}|=1$.

Since G is primitive, by the same argument that was used in the proof of Theorem 1, there exists $g_0\in G$ such that $Y_{K_0}\neq(Y_{K_0})g_0$ and $Y_{K_0}\cap(Y_{K_0})g_0\neq\emptyset$. Also, if $C_0=Y_{K_0}\cap(Y_{K_0})g_0$, $A_0=Y_{K_0}\setminus C_0$, and $B_0=(Y_{K_0})g_0\setminus C_0$, then (i) $Y_{K_0}=C_0\cup A_0$, $(Y_{K_0})g_0=C_0\cup B_0$, (ii) A_0, B_0 and C_0 are pairwise disjoint, (iii) $|A_0|=|B_0|\leqq|X\setminus Y_{K_0}|$, (iv) A_0, B_0, C_0 are all nonvoid, and (v) $|X\setminus[Y_{K_0}\cup(Y_{K_0})g_0]|=|X\setminus Y_{K_0}|-r$, where $r=|A_0|\geqq 1$ and is finite.

By Proposition 2, $\{K_0, K_0^{g_0}\}$ is transitive on $W = Y_{K_0} \cup (Y_{K_0})g_0$ (hence $|W| \geqq 2$) and fixes the elements of $X \backslash W$. But $|X \backslash W| = |X \backslash Y_{K_0}| - r < |X \backslash Y_{K_0}|$. Hence (by the minimality of $|X \backslash Y_{K_0}|$) it must be that $X \backslash W = \emptyset$, i.e., $X = W$.

We can write $A_0 = \{a_1, \cdots, a_r\}$, $B_0 = \{b_1, \cdots, b_r\}$, $Y_{K_0} = C_0 \cup \{a_1, \cdots, a_r\}$, $(Y_{K_0})g_0 = C_0 \cup \{b_1, \cdots, b_r\}$, and $X = C_0 \cup \{a_1, \cdots, a_r\} \cup \{b_1, \cdots, b_r\}$. We also observe that $|Y_{K_0}| > r = |X \backslash Y_{K_0}|$.

Suppose $r > 1$, i.e., $|X \backslash Y_{K_0}| > 1$. Then since G is primitive there is some $g \in G$ which maps some "b" to a "b," say b_i to b_j, and some "b" to an "a" or "c" ($c \in C_0$). Since the number of "b"s is finite, not all "b"s can be images of "b"s under g. Hence g must map some "a" or "c" to a "b," say to b_k. Further, since $r < |Y_{K_0}|$, not every "a" and "c" can be mapped onto a "b" by g. Therefore some "a" or "c" is mapped by g onto an "a" or "c," i.e., $Y_{K_0} \cap (Y_{K_0})g \neq \emptyset$; furthermore, $b_k \in (Y_{K_0})g \backslash Y_{K_0}$ and $b_j \notin Y_{K_0} \cup (Y_{K_0})g = V$. Hence $Y_{K_0} \neq V$, $|V| \geqq 2$ and $0 \neq |X \backslash V| < |X \backslash Y_{K_0}|$. But by Proposition 2, $\{K_0, K_0^g\}$ is transitive on V and fixes $X \backslash V$. Hence, by the minimality of $|X \backslash Y_{K_0}|$, $|X \backslash Y_{K_0}| \leqq |X \backslash V|$, a contradiction, so that $|X \backslash Y_{K_0}| = 1$, and G is doubly transitive on X.

Reference

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