ON A FACTORISATION OF FREE MONOIDS

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A property is given which relates two results of Spitzer [6]; it also relates two results of Chen, Fox and Lyndon [1]; the same remark applies to work of Meyer-Wunderli [5] and M. Hall [3] and to its generalisation by Lazard [4]. These connections are indicated more fully below.

In what follows, F is the free monoid generated by a fixed set X and F^+ denotes the set of all words of positive length of F. If the words f and f' of F belong to a submonoid F' of F, the words ff' and $f'f$ are said to be *F'-conjugate.* We consider the following conditions, I, I′ and II, on a family $\{Y_j: j\in J\}$ of subsets of F^+ indexed by a totally ordered set J.

(I) (resp. (I′)). Each $f\in F^+$ has at most (resp. at least) one representation in the form $f=f_1f_2\cdots f_n$, $n>0$, where each $f_i\in Y_{j_i}$ and $j_1\geqq j_2\geqq\cdots\geqq j_n$.

(II) Each F-conjugate class C has nonempty intersection with the submonoid F_j generated by Y_j for exactly one $j\in J$; further, $C\cap F_j$ is an F_j-conjugate class.

PROPOSITION 1. *Any two of the three conditions* I, I′ *and* II *imply the third one.*

PROOF. Let $\mathfrak{A}$ be the large algebra of F over the real field R. If U is a subset of F, we write $\boldsymbol{U}=\sum\{f: f\in U\}\in\mathfrak{A}$. Since $(1-\boldsymbol{U})^{-1}=1+\sum\{\boldsymbol{U}^m: m>0\}$, it follows that $(1-\boldsymbol{U})^{-1}=\boldsymbol{G}$ iff G is a submonoid freely generated by U.

Let us assume first that I and I′ are satisfied; it follows that each F_j, $j\in J$, is freely generated by Y_j and that $(1-\boldsymbol{X})^{-1}=\prod\{(1-\boldsymbol{Y}_j)^{-1}: j\in J\}$ where the product is taken according to the given ordering of J. Further, $\operatorname{Log}(1-\boldsymbol{X})^{-1}=\sum\{m^{-1}\boldsymbol{X}^m: m>0\}=\sum\{(\lambda f)^{-1}f: f\in F^+\}$ and $\operatorname{Log}(1-\boldsymbol{Y}_j)^{-1}=\sum\{(\lambda_j f)^{-1}f: f\in F^+\cap F_j\}$, where λf (resp. $\lambda_j f$) denotes the length of the word f with respect to the free basis X (resp. Y_j).

For each F-conjugate class C, let π_C denote the linear map of $\mathfrak{A}$ onto R that satisfies $\pi_C f=1$ if $f\in C$ and $\pi_C f=0$ if $f\in F\backslash C$. Since π_C is constant on conjugate classes, for all $f', f''\in F$ we have $\pi_C(f'f'')=\pi_C(f''f')$; it follows that if $\mathfrak{L}\subset\mathfrak{A}$ is the large Lie algebra over R generated by F, then $\pi_C\mathfrak{L}'=0$ for $\mathfrak{L}'=[\mathfrak{L},\mathfrak{L}]$. According to our hy-

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pothesis, $\text{Log}(1-\boldsymbol{X})^{-1}=\text{Log}\prod(1-\boldsymbol{Y}_j)^{-1}$ whence, by the Campbell-Hausdorff formula $\text{Log}(1-\boldsymbol{X})^{-1}=\sum\{\text{Log}(1-\boldsymbol{Y}_j)^{-1}:\ j\in J\}+K$ where $K\in\mathfrak{L}'$. Consequently,

$$(1)\qquad \pi_C\,\text{Log}(1-\boldsymbol{X})^{-1}=\sum\{\pi_C\,\text{Log}(1-\boldsymbol{Y}_j)^{-1}:j\in J\}.$$

If $f\in C$ has the form $f=g^p$ with maximal positive p, it follows that p Card $C=\lambda C$ where λC is the common length of all $f\in C$; in particular, p is independent of the choice of $f\in C$. Now $\pi_C\,\text{Log}(1-\boldsymbol{X})^{-1}=\sum\{(\lambda f)^{-1}:f\in C\}=(\lambda C)^{-1}\text{ Card }C=p^{-1}$. From (1) we conclude that

$$(2)\qquad p^{-1}=\sum\{(\lambda_j(C\cap F_j))^{-1}\text{ Card }(C\cap F_j):j\in J\}.$$

If $p=1$, the sum in (2) can have only one nonzero term and II is verified for C. If $p>1$, we conclude from the case $p=1$ that g has an F-conjugate $g'\in F_{j_0}$ for some $j_0\in J$. It follows that $f'=g'^p\in C\cap F_{j_0}$, that $C\cap F_{j_0}$ is an F_{j_0}-conjugate class and that $(\lambda_{j_0}(C\cap F_{j_0}))^{-1}$ Card $(C\cap F_{j_0})=p^{-1}$. It now follows from (2) that $C\cap F_j\neq\varnothing$ iff $j=j_0$, and the implication I & I'$\Rightarrow$II is verified.

Let us assume now that II is satisfied; it follows that for each $j\in J$ one has

$$(3)\qquad \text{for any } f,f'\in F,\text{ if } ff',f'f\in F_j,\text{ then } f,f'\in F_j.$$

Consequently (cf., e.g., [2]), each F_j is freely generated by Y_j and (2), whence (1), holds for every F-conjugate class C. Let α be the natural homomorphism of $\mathfrak{A}$ into the large algebra over R of the free commutative monoid generated by X. We deduce from (1) that $\alpha\,\text{Log}(1-\boldsymbol{X})^{-1}=\sum\{\alpha\,\text{Log}(1-\boldsymbol{Y}_j)^{-1}:j\in J\}$, or, in equivalent fashion that $\alpha(1-\boldsymbol{X})^{-1}=\alpha\prod\{(1-\boldsymbol{Y}_j)^{-1}:j\in J\}$. Now, I (resp. I') is equivalent to $S+\prod(1-\boldsymbol{Y}_j)^{-1}=(1-\boldsymbol{X})^{-1}$ where S (resp. $-S$) is an element of $\mathfrak{A}$ in which every $f\in F$ has non-negative coefficient. Since $\alpha S=0$ implies $S=0$, the implication I & II$\Rightarrow$I' (resp. I' & II$\Rightarrow$I) is verified.

EXAMPLE 1. Let σ be a homomorphism of F into the additive group of R and identify J with R. For $r\in R$, let Y_r be the set of all $f\in F^+$ such that $\sigma f=r\lambda f$ and that $\sigma f'<r\lambda f'$ for every factorisation $f=f'f''$ ($f'\neq 1, f$). The fact that $\{Y_r: r\in R\}$ satisfies I and I' (resp. II) is proved by Spitzer in [6, p. 327] (resp. p. 324).

EXAMPLE 2. Let $\leqq$ denote a lexicographic order on F and let J be the set H of all $f\in F^+$ such that $f=f'f''$ for $f',f''\in F^+$ implies $f<f''f'$. Let $Y_h=\{h\}$, for each $h\in H$. The fact that I, I' and II are satisfied is due to Chen, Fox and Lyndon [1] (cf. also [7]). A similar result holds when H is replaced by the set obtained by "removing the brackets" from Hall's *basic commutators* ([5] and [3, Chapter 11]).

We conclude with the following application of the "elimination method" of Lazard [4].

PROPOSITION 2. *Let F be a free monoid, and P_1 and P_2 two subsets of F such that $\boldsymbol{F}^+=\boldsymbol{P}_1+\boldsymbol{P}_2$. Then there exists a unique pair of subsets $Y_1\subset P_1$ and $Y_2\subset P_2$ such that*

$$\boldsymbol{F} = (1 - \boldsymbol{Y}_1)^{-1}(1 - \boldsymbol{Y}_2)^{-1}. \tag{4}$$

PROOF. Let X be a free set of generators of F and let $X_{i,0}=X\cap P_i$ $(i=1, 2)$. Then $\boldsymbol{W}_0=(1-\boldsymbol{X}_{2,0})^{-1}\boldsymbol{X}_{2,0}\boldsymbol{X}_{1,0}(1-\boldsymbol{X}_{1,0})^{-1}$ is the sum of all $f=f_2f_1$ where f_1 is a nontrivial word in the elements of $X_{1,0}$ and f_2 in those of $X_{2,0}$. It follows that $\boldsymbol{F}=(1-\boldsymbol{X}_{1,0})^{-1}(1-\boldsymbol{W}_0)^{-1}(1-\boldsymbol{X}_{2,0})^{-1}$. If we let $Y_{i,0}=X_{i,0}$ $(i=1, 2)$ this establishes for $k=0$ the inductive hypothesis that

$$X_{i,k} \subset Y_{i,k} \subset P_i \quad (i = 1, 2) \quad \text{and} \quad \boldsymbol{F} = \boldsymbol{F}_{1,k}(1 - \boldsymbol{W}_k)^{-1}\boldsymbol{F}_{2,k} \tag{5}$$

where

$$\boldsymbol{F}_{i,k} = (1 - \boldsymbol{Y}_{i,k})^{-1} \quad (i = 1, 2) \quad \text{and} \quad \boldsymbol{W}_k = \boldsymbol{F}_{2,k}\boldsymbol{X}_{2,k}\boldsymbol{X}_{1,k}\boldsymbol{F}_{1,k}.$$

Suppose (5) is satisfied for some $k\geqq 0$. We construct inductively a sequence of subsets $W_{k,n}$ of W_k for all $n\geqq 0$. First we take $W_{k,0}=\emptyset$. Supposing $W_{k,n}$ given we define $W_{k,n+1}$ to be the union of $W_{k,n}$ with the set of all words of minimal length in the complement of $(W_{k,n}\cap P_1)F_{1,k}\cup F_{2,k}(W_{k,n}\cap P_2)$ in W_k. We now define

$$X_{i,k+1} = \bigcup_{n\geq 0} (W_{k,n} \cap P_i); \qquad Y_{i,k+1} = Y_{i,k} \cup X_{i,k+1} \qquad (i = 1, 2).$$

Thus, $X_{i,k+1}\subset Y_{i,k+1}\subset P_i$ $(i=1, 2)$. To complete the verification that (5) holds for $k+1$, we need to show first

$$\boldsymbol{W}_k = \boldsymbol{X}_{1,k+1}\boldsymbol{F}_{1,k} + \boldsymbol{F}_{2,k}\boldsymbol{X}_{2,k+1}. \tag{6}$$

Indeed, by the inductive hypothesis each $f\in W_k$ has a unique representation in the form $f=f_2f_1$, where $f_1\in X_{1,k}F_{1,k}$ and $f_2\in F_{2,k}X_{2,k}$. Taking $W_k=F_{2,k}W_kF_{1,k}$ into account, it follows that there exist two sets T_1 and T_2 such that $\boldsymbol{T}_1 = \boldsymbol{X}_{1,k+1}\boldsymbol{F}_{1,k}$, $\boldsymbol{T}_2 = \boldsymbol{F}_{2,k}\boldsymbol{X}_{2,k+1}$, and $W_k=T_1\cup T_2$. Thus the proof of (6) needs only the verification that $T_1\cap T_2=\emptyset$.

Let $f\in T_2$. By definition $f=g_2f_2f_1$, where $g_2\in F_{2,k}$, $f_2\in F_{2,k}X_{2,k}$, $f_1\in X_{1,k}F_{1,k}$, and $f_2f_1\in W_{k,n}\cap P_1$ for some $n\geqq 0$. The definition of $W_{k,n}$ implies that $f_2f_1\notin X_{1,k+1}F_{1,k}$. Thus, for each $n'\geqq 0$ and for each left factor $f_1'\in X_{1,k}F_{1,k}$ of f_1, we have $f_2f_1'\notin W_{k,n'}\cap P_1$. It follows that for each such f_1' we have $f_2f_1'\in T_2$, hence $g_2f_2f_1'\in T_2$, and finally $g_2f_2f_1'$ $\notin W_{k,n''}\cap P_1$ for all $n''\geqq 0$. This shows that $f=g_2f_2f_1\notin T_1$ and $T_1\cap T_2=\emptyset$, hence (6) is proved.

For the rest, we compute as follows:

$$(F_{1,k+1}(1 - W_{k+1})^{-1}F_{2,k+1})^{-1}$$
$$= (1-Y_{2,k+1})(1-(1-Y_{2,k+1})^{-1}X_{2,k+1}X_{1,k+1}(1-Y_{1,k+1})^{-1})(1-Y_{1,k+1})$$
$$= 1-Y_{2,k+1}-Y_{1,k+1}+Y_{2,k+1}Y_{1,k+1}-X_{2,k+1}X_{1,k+1}$$
$$= 1-(Y_{2,k}+X_{2,k+1})-(Y_{1,k}+X_{1,k})+(Y_{2,k}+X_{2,k+1})(Y_{1,k}+X_{1,k+1})$$
$$-X_{2,k+1}X_{1,k+1}$$
$$= 1-Y_{2,k}-Y_{1,k}+Y_{2,k}Y_{1,k}-(1-Y_{2,k})X_{1,k+1}-X_{2,k+1}(1-Y_{1,k})$$
$$= (1-Y_{2,k})(1-X_{1,k+1}(1-Y_{1,k})^{-1}-(1-Y_{2,k})^{-1}X_{2,k+1})(1-Y_{1,k})$$
$$= F_{2,k}^{-1}(1-W_k)F_{1,k}^{-1}=F^{-1}$$

Finally, since $W_0 \subset FXXF$ and $W_{k+1} \subset FW_kW_kF$, each W_k ($k \geqq 0$) contains no word of length less than 2^{k+1}. It follows that the same is true for the set complement of $(1-Y_{1,k})^{-1}(1-Y_{2,k})^{-1}$ in F. Thus, letting $Y_i = \bigcup_{k \geq 0} Y_{i,k}$ ($i=1, 2$), we have proved the existence of at least one pair of sets satisfying the conditions stated in Proposition 2.

To verify the uniqueness, let us consider any other pair of subsets Y_1' and Y_2' of F^+ that satisfies $F=(1-Y_1')^{-1}(1-Y_2')^{-1}$. We define the subsets U_i, V_i, V_i' ($i=1, 2$) of F^+ by the relations $U_i = Y_i \cap Y_i'$; $V_i = Y_i - U_i$; $V_i' = Y_i' - U_i$ ($i=1, 2$). From $F^{-1}=(1-Y_2)(1-Y_1)$ $=(1-Y_2')(1-Y_1')$ we deduce

$$-V_2 - V_1 + U_2V_1 + V_2U_1 = - V_1' - V_2' + U_2V_1' + V_2'U_1. \tag{7}$$

Now, either $Y_1 = Y_1'$ and $Y_2 = Y_2'$ (i.e., $V_1 \cup V_2 \cup V_1' \cup V_2' = \emptyset$) or else $V_1 \cup V_2 \cup V_1' \cup V_2'$ contains some element f of minimal length. By construction $V_1 \cap V_1' = V_2 \cap V_2' = \emptyset$. Thus (7) shows that $f \in (V_1 \cap V_2')$ $\cup (V_2 \cap V_1')$. Since this last set is empty if $Y_1' \subset P_1$ and $Y_2' \subset P_2$, the verification of Proposition 2 is concluded.

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