SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

AN EVEN SIMPLER PROOF OF OPIAL'S INEQUALITY

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The inequality

$$2\int_0^a |y'(x)y(x)|\,dx \leqq a\int_0^a |y'(x)|^2\,dx,$$

valid for $y(x)$ absolutely continuous with $y(0)=0$, has received successively simpler proofs at the hands of Opial [4], Olech [3], Beesack [1], and Levinson [2]. The following proof is conjectured to attain the ultimate in simplicity.

Define

$$z(x) = \int_0^x |y'(u)|\,du, \qquad 0 \leqq x \leqq a.$$

Then for $0\leqq x\leqq a$, $|y(x)| \leqq z(x)$, and we have

$$2\int_0^a |y'(x)y(x)|\,dx \leqq 2\int_0^a z'(x)z(x)\,dx = z^2(a).$$

Now by Schwarz,

$$z^2(a) = \left\{\int_0^a z'(x)\,dx\right\}^2 \leqq \int_0^a dx\int_0^a (z'(x))^2\,dx = a\int_0^a |y'(x)|^2\,dx.$$

There is equality only if $y=bx$ with b constant.

References

1. Paul R. Beesack, *On an integral inequality of Z. Opial*, Trans. Amer. Math. Soc. **104** (1962), 470–475.

2. N. Levinson, *On an inequality of Opial and Beesack*, Proc. Amer. Math. Soc. **15** (1964), 565–566.

3. C. Olech, *A simple proof of a certain result of Z. Opial*, Ann. Polon. Math. **8** (1960), 61–63.

4. Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8** (1960), 29–32.

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