SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

AN EVEN SIMPLER PROOF OF OPIAL'S INEQUALITY

C. L. MALLOWS

The inequality

\[ 2 \int_0^a |y'(x)y(x)| \, dx \leq a \int_0^a |y'(x)|^2 \, dx, \]

valid for \( y(x) \) absolutely continuous with \( y(0) = 0 \), has received successively simpler proofs at the hands of Opial [4], Olech [3], Beesack [1], and Levinson [2]. The following proof is conjectured to attain the ultimate in simplicity.

Define

\[ s(x) = \int_0^x |y'(u)| \, du, \quad 0 \leq x \leq a. \]

Then for \( 0 \leq x \leq a \), \( |y(x)| \leq s(x) \), and we have

\[ 2 \int_0^a |y'(x)y(x)| \, dx \leq 2 \int_0^a s'(x)s(x) \, dx = s^2(a). \]

Now by Schwarz,

\[ s^2(a) = \left\{ \int_0^a s'(x) \, dx \right\}^2 \leq \int_0^a dx \int_0^a (s'(x))^2 \, dx = a \int_0^a |y'(x)|^2 \, dx. \]

There is equality only if \( y = bx \) with \( b \) constant.

REFERENCES


BELL TELEPHONE LABORATORIES

Received by the editors August 22, 1964

173