A CONVOLUTION EQUATION ON A COMPACT INTERVAL

P. E. GREENWOOD[1]

1. **Introduction.** We present two theorems which give sufficient conditions for the existence of a solution, s, of the convolution equation $g(x)=\int_J W(x-y)\,ds(y)$ (briefly, $g=W*s$). Here J is, say, the compact real interval $[0, 1]$, g is an $L^2(J, dx)$ function, and W is a complex-valued function on $[-1, 1]$ with the following property: The operator $[D^2W*]: f\to D^2W*f = \int_{-1}^{1} f(x-y)\,dD^2W(y)$ maps $L^2(J, dx)$ into itself, where differentiation, D, and convolution, $*$, are distribution operators as defined by L. Schwartz [1]. The guaranteed solution, s, will be a distribution with support in J, of order depending on the differentiability of W.

In recent treatments of the convolution equation over a compact interval by Shinbrot and Akutowicz [2], [3], [4], hypotheses are made on the Fourier transform of W, $\mathfrak{F}W$, which place restrictions on the real zeros of $\mathfrak{F}W$ and exclude such kernels as $\mathfrak{F}^{-1}\sin^2 x/x^2$. Now, the values of W outside a neighborhood of $[-1, 1]$ do not influence $W*s$ on J if the distribution s has support in J. The existence and nature of solutions of the convolution equation depends in principle, therefore, only upon W as it is defined in a neighborhood of $[-1, 1]$. Although in this paper we do not strictly generalize the results of [2], [3], [4], we do, in effect, restrict the values of W only on a neighborhood of $[-1, 1]$.

We employ Chover's idea [5], of assuming W not differentiable at 0. D^2W then has a point mass at 0, and the operator $[D^2W*]$ can be written as the sum of two operators, one of which is the identity, I. In our theorems we assume that $D^2W=k-\delta_0-\psi$ on $[-1, 1]$, where k is a function of bounded variation and ψ is a measure of variation less than 1. Results for kernels W such that D^nW has this form for some positive integer n follow immediately as corollaries.

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2. **Definitions.** We use the term positive definite in the following sense. We call a continuous complex-valued function W on the real

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line, R, *positive definite* if for every nonzero Radon measure m of compact support on R, $\int\int W(x-y)\,dm(x)d\bar{m}(y)>0$. We use the equivalent form, $W * m * \tilde{\bar{m}}(0)>0$, where $\tilde{\bar{m}}$ denotes the reflection of $\bar{m}$ about 0.

In expressions involving the distribution operators D and $*$, we regard functions and measures as the associated distributions. We use several theorems found in [1] without specific reference. The primary one is that if f and g are distributions, at least one of which has compact support, then $D(f*g)=Df*g=f*Dg$.

We regard functions in $L^2(J, dx)$ as identically 0 outside of J. δ_x denotes the measure which is 0 except at x where it has mass 1. Hence $[\delta_0 *]$ is the identity operator, I. By $[D^2W*]$ we mean the operator on $L^2(J, dx)$ which maps a function f into the function which is D^2W*f on J and zero elsewhere. If $H\subset J$, $J-H=\{x: x\in J$ and $x\notin H\}$. If T is an operator, $R(T)=$range of T, and $N(T)=$null space of T.

In Theorem 2 there appear distributions which are functions on the interior of J and are extendable to functions of bounded variation on J. We call the class of such distributions B.V.(J).

3. Existence theorems and proofs.

THEOREM 1. *Let W be a positive definite function on $[-1, 1]$ such that $D^2W=k-\delta_0-\psi$, where k is a function of bounded variation on $[-1, 1]$ and ψ is a measure on $[-1, 1]$ of variation less than 1. Then for any $g\in L^2(J)$ there exists $f\in L^2(J)$ such that $g=W*D^2f$ on J. If g is a continuous function of bounded variation on J then f is also.*

In the proof we use the following lemma.

LEMMA 1. *If $g=k*f-f-\psi*f$ on J, where $f\in L^2(J)$ and g is a continuous function of bounded variation on J, then f is a continuous function of bounded variation on J.*

PROOF. Since the convolution of two L^2 functions is continuous and k has bounded variation, $k*f$ is a continuous function of bounded variation on J and so is $k*f-g=f+\psi*f$. As is well known, var $\psi<1$ implies that $[\psi*]$ is an operator of norm <1 on $L^2(J)$ as well as on C_J. There exists $f_1\in C_J$ such that $f_1+\psi*f_1=f+\psi*f$ on J. Since f_1 and f are both in $L^2(J)$, $f_1=f$ and $f\in C_J$.

To show that f is a function of bounded variation on J it suffices to show that $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$ is uniformly bounded for any n and for any choice of $\{x_i, i=0, \cdots, n\}\subset J$. Let $v=\text{var}_J(f+\psi*f)$. Fix n and choose $\{x_i, i=0, \cdots, n\}\subset J$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is maximized. Then

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &\leqq v + \sum_{i=1}^{n} \int_{-1}^{1} |f(x_i - y) - f(x_{i-1} - y)| \; d\psi(y) \\ &\leqq v + \operatorname{var} \psi \cdot \sup_y \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)|.\end{aligned}$$

Since $g=0$ on the complement of J,

$$\begin{aligned}\sup_y \sum_{i=1}^{n} |f(x_i - y) - f(x_{i-1} - y)| \leqq \max_{\{z_i\}\subset J} &\sum_{i=1}^{n} |f(z_i) - f(z_{i-1})| \\ &+ \sup |f| .\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leqq (v + \operatorname{var} \psi \cdot \sup |f|)/(1 - \operatorname{var} \psi).$$

This completes the proof of the lemma.

To prove Theorem 1 we first note that since W is positive definite, it is Hermitian, and $[D^2W *] = [(k-\delta_0-\psi) *]$ is a self-adjoint operator on $L^2(J)$. The closure of $R[D^2W *]$ is therefore the orthogonal complement of $N[D^2W *]$. We prove the first statement of the theorem by showing that $R[D^2W *]$ is closed and that $N[D^2W *]=0$. Then $R[D^2W *]=L^2(J)$ and for any $g\in L^2(J)$ there exists $f\in L^2(J)$ such that $g=D^2W * f= W * D^2f$.

If T and S are continuous linear operators on L^2 such that T is completely continuous and S^{-1} is continuous then $R[T-S]$ is closed. This can be shown by an argument closely analogous to that for the familiar case where S is the identity. Since $k(x-y)$ is an element of $L^2([-1, 1]\times[-1, 1])$, $[k *]$ is a completely continuous operator on $L^2(J)$. $[(\delta_0+\psi) *]=I+[\psi *]$ has a continuous inverse. Therefore $R[[k *]-[(\delta_0+\psi) *]]$ is closed.

Suppose $f\in N[D^2W *]$, i.e., $k * f-f-\psi * f=0$ on J. By Lemma 1, Df is a measure on R. Since W is positive definite, $W * Df * (Df)^{\sim}(0) > 0$ unless $Df\equiv 0$. $W * Df * (Df)^{\sim}(0) = -(D^2W * f) * \bar{f}(0) = -\int_{-J}(D^2W * f)(-y)\bar{f}(-y)dy$ since the support of $\bar{f}$ is contained in $-J$. But since $D^2W * f=0$ on J, $W * Df * (Df)^{\sim}(0)=0$, and $Df\equiv 0$. f has support in J. Therefore $Df\equiv 0$ implies $f\equiv 0$.

Suppose g is a continuous function of bounded variation on J, and let f be the element of $L^2(J)$ such that $g=D^2W * f=k * f-f-\psi * f$.

By Lemma 1, f is a continuous function of bounded variation on J.

From this theorem we get, immediately:

COROLLARY 1. *If $D^n W$ satisfies the hypothesis of Theorem 1, i.e., $D^n W$ is a positive definite function on $[-1, 1]$ such that $D^{n+2}W = k - \delta_0 - \psi$ as stated, then for any $g \in L^2(J)$ there exists $f \in L^2(J)$ such that $g = W * D^{n+2}f$. If g is a continuous function of bounded variation on J, then $D^{n+2}f$ is an $(n+1)$st order distribution with support in J.*

THEOREM 2. *Let H be a closed interval contained in the interior of J. Let T be the self-adjoint operator on $L^2(J)$ defined by $Tf = (k - \delta_0) * f$, where k is a complex-valued, Hermitian function of bounded variation on $[-1, 1]$. Then*

(a) *For any $g \in L^2(H)$ there exists $f \in L^2(J)$ such that $g = Tf$ on H.*

(b) *If g is a function of bounded variation on H, there exists $f \in L^2(J)$ of bounded variation such that $g = Tf$ on H.*

(c) *If Dg is in $B.V.(H)$, there exists $f \in L^2(J)$ such that Df is in $B.V.(J)$ and $g = Tf$ on H.*

In the proof of Theorem 2 we use the following lemma.

LEMMA 2. *If $g = k * f - f$ on J, where Dg is in $B.V.(J)$ and f is a measure with support in J then Df is in $B.V.(J)$.*

PROOF. Since $k * f$ and g are both functions of bounded variation on J, f is a function of bounded variation on J and therefore on R. Hence, Df is a measure, and $D(k * f) = k * Df$ is in B.V.(J). Therefore so is Df, which proves the lemma.

To prove Theorem 2 we first note that T is a self-adjoint operator on $L^2(J)$ and that $R(T)$ is closed and is the orthogonal complement of $N(T)$. Given $g \in L^2(H)$, we will construct an element of the orthogonal complement of $N(T)$ which equals g on H.

Since $[k *]$ is a completely continuous operator on $L^2(J)$, $N(T)$ is finite-dimensional. Choose $\{f_i, i = 1, \cdots, n\}$, linearly independent on J and spanning $N(T)$. Then the f_i are linearly independent on $J - H$, as we now show.

Consider any $f = \sum_{i=1}^n c_i f_i$. $f \neq 0$ on J. Suppose $f = 0$ on $J - H$. Then $f = 0$ on $H^c = R - H$. f is an element of $N(T)$, i.e., $k * f - f = 0$ on J. Lemma 2, where $g = 0$, implies that Df is in B.V.(J). Therefore $D^2 f$ is a measure on the interior of J. $D^2 f = 0$ on H^c. Therefore $D^2 f$ is a measure on R, with support in J. By Lemma 2, again, $D^3 f$ is in B.V.(J). $D^4 f = 0$ on H^c. Therefore $D^4 f$ is a measure on R. Similarly, $D^{2n} f$ is a measure on R implies $D^{2n+2} f$ is a measure on R, and so by induction $D^n f$ is a measure for all n. For any distribution h, Dh is a function implies h is a continuous function, and Dh is a continuous function implies h is differentiable and $Dh=h^{(1)}$. Therefore $D^n f=f^{(n)}$, $f\in C^\infty$, and $f^{(n)}\in L^2(J)$ for all n. $f^{(n)}$ is, consequently, an element of the domain of T, and $Tf^{(n)}=TD^n f=(k-\delta_0)*D^n f=D^n[(k-\delta_0)*f]$, which is 0 on the interior of J because $(k-\delta_0)*f=0$ on J. Since f is in C^∞ on all of R, so is $D^n[(k-\delta_0)*f]$. Therefore $Tf^{(n)}=0$ on J, i.e., $f^{(n)}\in N(T)$ for all n. But since $N(T)$ is finite-dimensional, there exists an integer m and constants b_j, $j=0,\cdots,m$, not all zero, such that $\sum_{j=0}^m b_j f^{(j)}=0$ on J. Consequently f is analytic on the interior of J and cannot be $=0$ on $J-H$. It follows that the f_i are linearly independent on $J-H$ as asserted.

The f_i can be chosen so that their restrictions to $J-H$ form an orthonormal set. For instance, if $g_i=f_i-\sum_{j=1}^{i-1}a_{ij}g_j$, $i=1,\cdots,n$, is the Gram-Schmidt orthogonalization of the f_i restricted to $J-H$, then the g_i considered as functions on J still span $N(T)$, since each f_i is a linear combination of the g_i. We assume that the f_i are so chosen.

Define

$$G(x)=g(x)\ \text{for}\ x\in H,\qquad G(x)=\sum_{i=1}^n b_i f_i(x)\ \text{for}\ x\in J-H,$$

where $b_i=-\int_H f_i(x)g(x)\,dx$. Then $G\in L^2(J)$.

$$\int_J f_i(x)G(x)\,dx=\int_{J-H} f_i(x)\left(\sum_{j=1}^n b_j f_j(x)\right)dx+\int_H f_i(x)g(x)\,dx$$
$$=-b_i+b_i=0,$$

for each $i=1,\cdots,n$. Therefore G is in the orthogonal complement of $N(T)$ which as noted above is $R(T)$, i.e., there exists $f\in L^2(J)$ such that $G=Tf$ on J. Since $g=G=Tf$ on H, (a) is proved.

(b) Suppose g is a function of bounded variation on H. If $f_i\in N(T)$, i.e., $k*f_i=f_i$ on J, then f_i is a function of bounded variation. Therefore G, as defined above, is a function of bounded variation, and $Tf=k*f-f=G$ on J implies f is a function of bounded variation on J and therefore on R.

(c) Suppose Dg is in B.V.(H). If $f_i\in N(T)$ then Lemma 2 implies that Df_i is in B.V.(J). Therefore G as defined above is such that DG can fail to be in B.V.(J) only if G has discontinuities at the endpoints, c and d, of H. To remedy this we will alter the definition of G on $J-H$. Let $h(x)$ be a function on $J-H$ such that

(i) h is in the orthogonal complement of $N(T)$,

(ii) Dh is a function of bounded variation on $J-H$,

(iii) $h(c)=g(c)-\sum_{i=1}^n b_i f_i(c)$, $h(d)=g(d)-\sum_{i=1}^n b_i f_i(d)$.

Define

$$G(x) = \sum_{i=1}^{n} b_i f_i(x) + h(x) \text{ for } x \in J - H, \qquad G(x) = g(x) \text{ for } x \in H.$$

Then DG is in B.V.(J), and G is in the orthogonal complement of $N(T)$. There exists $f \in L^2(J)$ such that $G = k * f - f$ on J. Lemma 2 implies that Df is in B.V.(J).

We obtain immediately from Theorem 2.

COROLLARY 2. *If W is a function on $[-1, 1]$ such that $D^n W = k - \delta_0$ for some positive integer n, where k is as in Theorem 2, then for every $g \in L^2(H)$ there exists $f \in L^2(J)$ such that $g = W * D^n f$ on H.*

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UNIVERSITY OF WISCONSIN