CONTINUITY AND LOCATION OF ZEROS OF LINEAR COMBINATIONS OF POLYNOMIALS

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1. **Introduction.** It is well known (see Marden [1, p. 3], Specht [4, p. 13], Ostrowski [2]) that the zeros of a polynomial $\sum_0^n a_k z^k$ vary continuously when all its coefficients, with the exception of a_n, are varied continuously. The reason for this exception is that the situation is altered when the leading coefficient a_n takes the value 0. Theorem 1 is a formulation of a continuity principle where the leading coefficients are allowed to vanish and Theorems 3 and 4 apply this principle to the location of zeros of linear combinations of two polynomials. Theorem 4 improves on a recent result of Rubinstein [3]. Theorems 6 and 7 specify disks containing the zeros of a linear combination of a finite number of polynomials whose zeros lie in prescribed disks.

2. **The continuity of zeros.** The first theorem asserts that if the coefficients of a polynomial, some of whose leading coefficients are zero, are varied continuously, the existing zeros of the polynomial vary continuously whereas any new zeros emerge from a neighborhood of the point at infinity.

The coefficients of all polynomials in the following are complex numbers. $D(c, R)$ denotes the closed disk $|z-c| \leq R$.

THEOREM 1. *Given a polynomial* $p_n(z) \equiv \sum_0^n a_k z^k$, $a_n \neq 0$, *an integer* $m \geq n$ *and a number* $\epsilon > 0$, *there exists a number* $\delta > 0$ *such that whenever the* $m+1$ *complex numbers* b_k, $0 \leq k \leq m$, *satisfy the inequalities*

(1) $|b_k - a_k| < \delta$ *for* $0 \leq k \leq n$ *and, if* $m > n$, $|b_k| < \delta$ *for* $n+1 \leq k \leq m$,

then the zeros β_k, $1 \leq k \leq m$, *of the polynomial* $q_m(z) \equiv \sum_0^m b_k z^k$ *can be labeled in such a way as to satisfy with respect to the zeros* α_k, $1 \leq k \leq n$, *of* $p_n(z)$ *the inequalities*

(2) $|\beta_k - \alpha_k| < \epsilon$ *for* $1 \leq k \leq n$ *and, if* $m > n$, $|\beta_k| > 1/\epsilon$ *for* $n+1 \leq k \leq m$.

PROOF. We may assume that ϵ is sufficiently small so that the disks $D(\alpha_k, \epsilon)$, $1 \leq k \leq n$, are pairwise either identical or disjoint and that $D(0, 1/\epsilon)$ contains all the above disks. If C_k denotes the boundary of

Presented to the Society, January 23, 1964; received by the editors September 6, 1963.

[1] Research supported by the National Science Foundation Grant No. G-24469 with the University of Maryland.

$D(\alpha_k, \epsilon)$, $1 \leqq k \leqq n$, and C_0 is the boundary of $D(0, 1/\epsilon)$, then let $M = \min_{0 \leq k \leq n}(\min_{z \in C_k} |p_n(z)|)$. Obviously $M > 0$. We pick δ to satisfy $\delta(\epsilon^m + \cdots + \epsilon + 1) < M\epsilon^m$. If now the b_k's satisfy the inequalities (1), we have for $z \in \bigcup_0^n C_k$

$$| q_m(z) - p_n(z) | \leqq \sum_0^n | b_k - a_k | |z|^k + \sum_{n+1}^m | b_k | |z|^k \leqq \sum_0^m \delta(1/\epsilon)^k < M.$$

Applying Rouché's theorem to $q_m(z) = p_n(z) + [q_m(z) - p_n(z)]$ for each of the circles C_k, $0 \leqq k \leqq n$, we deduce that each of the disks $D(\alpha_k, \epsilon)$, $1 \leqq k \leqq n$, contains as many β_k's as α_k's, thus implying the first part of (2), and that the disk $D(0, 1/\epsilon)$ contains precisely n zeros of $q_m(z)$, which proves the second part of (2).

3. **Linear combinations of two polynomials.** Our purpose is to study the location of the zeros of the polynomial in z $F(z, \lambda) \equiv p_m(z) + \lambda q_n(z)$ when it is known that the zeros of the polynomials $p_m(z)$ and $q_n(z)$ are, respectively, in the disks $D(c_1, R_1)$ and $D(c_2, R_2)$ and λ is a constant. The special case $m = n$ was studied by Walsh [5]. We shall therefore assume for the moment $m \neq n$ and return to the special case in §4. We first quote a theorem of Walsh [5]:

THEOREM 2. *If the points $\alpha_1, \alpha_2, \cdots, \alpha_n$ lie in the closed disk $D(c, r)$, the equation in α*

$$(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) = (z - \alpha)^n$$

has at least one root in $D(c, r)$.

Next we prove a theorem about a special linear combination of polynomials:

THEOREM 3. *For a fixed λ, let $G(z, \lambda) \equiv (z - \alpha_1)^m + \lambda(z - \alpha_2)^n$, where m and n are arbitrary positive integers, $m > n \geqq 1$. Let $K = m^m |\alpha_2 - \alpha_1|^{m-n}/n^n(m-n)^{m-n}$.*

I. *If r_1 is the unique*[2] *positive root of the equation*

$$A(x) \equiv x^m - |\lambda| (x + |\alpha_2 - \alpha_1|)^n = 0, \tag{3}$$

then the m zeros of $G(z, \lambda)$ lie in the disk $D(\alpha_1, r_1)$.

II. *If $|\lambda| > K$, let r_2, r_3 be the two*[3] *positive roots of the equation*

$$B(x) \equiv |\lambda| x^n - (x + |\alpha_2 - \alpha_1|)^m = 0. \tag{4}$$

[2] By Descartes' rule of signs and $A(0) < 0$, $A(\infty) > 0$.

[3] Since $B(0) < 0$, $B(\infty) < 0$, by Descartes' rule of signs there are at most two positive roots. We note that $B(y) \geqq 0$ is equivalent to $|\lambda| \geqq (y + |\alpha_2 - \alpha_1|)^m / y^n \equiv R(y)$ and that $K = \min[R(y), 0 < y]$.

Then $D(\alpha_2, r_2)$ *contains exactly* n *zeros of* $G(z, \lambda)$ *while the annulus* $r_2 < |z| < r_3$ *contains none.*

III. *If* $|\lambda| = K$, *let* r_2 *be the unique positive* (*double*) *root of* (4). *Then at least* n *zeros of* $G(z, \lambda)$ *lie in* $D(\alpha_2, r_2)$, *and at most* n *zeros lie in its interior.*

REMARK. It is easy to see that if for some $y_1 > 0$ we have $A(y_1) > 0$, then $r_1 < y_1$ and if for some $y_2 > 0$ we have $B(y_2) > 0$, then $r_2 < y_2 < r_3$. Therefore r_1 and r_2 can be replaced by y_1 and y_2, respectively, in Parts I and II of Theorem 3. This would result in a weaker but more convenient proposition.

PROOF. I. The m zeros of $G(z, 0)$ are all located at α_1. According to Theorem 1, as we vary the values of the parameter continuously from 0 to λ, the same number of zeros of $G(z, \lambda)$ will lie in a disk $D(\alpha_1, y)$, $y > 0$, unless at least one of the zeros, say z', has crossed the boundary $|z - \alpha_1| = y$ for some value λ' of the parameter. But for such λ' we have the inequality

$$|\lambda'| = \left| -\frac{(z' - \alpha_1)^m}{(z' - \alpha_2)^n} \right| \geqq \frac{y^m}{(y + |\alpha_2 - \alpha_1|)^n}. \tag{5}$$

Therefore if, for a fixed y, λ satisfies the opposite inequality

$$y^m - |\lambda|(y + |\alpha_2 - \alpha_1|)^n > 0, \tag{6}$$

smaller values of λ will also satisfy (6) and thus the m zeros of $G(z, \lambda)$ are in the interior of $D(\alpha_1, y)$. Part I of our theorem follows from the observation that $y > r_1$ implies (6).

II. Set $\mu = 1/\lambda$ and $H(z, \lambda) \equiv G(z, \lambda)/\lambda = (z - \alpha_2)^n + \mu(z - \alpha_1)^m$. Again we observe that the n zeros of $H(z, 0)$ are all at α_2. By Theorem 1, for sufficiently small values of μ, the disk $D(\alpha_2, y)$, $y > 0$, would still contain exactly n zeros of $H(z, \mu)$, the remaining $m - n$ zeros lying in a neighborhood of ∞. If for some value μ' of the parameter, a zero, say z', of $H(z, \mu)$ is on the boundary of $D(\alpha_2, y)$, we would have

$$|\mu'| = \left| -\frac{(z' - \alpha_2)^n}{(z' - \alpha_1)^m} \right| \geqq \frac{y^n}{(y + |\alpha_2 - \alpha_1|)^m}. \tag{7}$$

Therefore, if for a fixed value of y, μ satisfies the opposite inequality

$$y^n - |\mu|(y + |\alpha_2 - \alpha_1|)^m > 0, \tag{8}$$

smaller values of μ will also satisfy (8), and thus $D(\alpha_2, y)$ will contain precisely n zeros of $H(z, \mu)$, hence of $G(z, \lambda)$. Inequality (8) can be rewritten as $B(y) > 0$ and it is seen easily that $r_2 < y < r_3$ implies $B(y) > 0$ and Part II follows immediately.

III. If $|\lambda| = K$, $B(r_2) = 0$ but for $0 < y \neq r_2$, $B(y) < 0$. However, if $|\lambda| > K$ (or $|\mu| < 1/K$) we have

$$r_2^n - |\mu| (r_2 + |\alpha_2 - \alpha_1|)^m > 0$$

which indicates that as the absolute value of the parameter grows from 0 to $1/K$, the n zeros of $H(z, 0)$ remain inside $D(\alpha_2, r_2)$ and the additional $m-n$ zeros, which appear when $\mu \neq 0$, remain outside $D(\alpha_2, r_2)$. Only when $|\mu|$ reaches the value $1/K$ can some of the zeros of either group actually lie on the boundary of $D(\alpha_2, r_2)$. This completes the proof of Theorem 3.

We now come to the main result of this section:

THEOREM 4. *Let* $f_m(z) \equiv z^m + a_{m-1}z^{m-1} + \cdots + a_0$ *and* $g_n(z) \equiv z^n + b_{n-1}z^{n-1} + \cdots + b_0$ *be two polynomials whose zeros lie, respectively, in the disks* $D(c_1, R_1)$ *and* $D(c_2, R_2)$ *and suppose* $m > n \geqq 1$. *For a fixed* λ *let* $F(z, \lambda) \equiv f_m(z) + \lambda g_n(z)$. *Then*:

I. *If* ρ_1 *is the unique positive root of the equation*

$$C(x) \equiv x^m - |\lambda| (x + |c_2 - c_1| + R_1 + R_2)^n = 0, \tag{9}$$

then the m zeros of $F(z, \lambda)$ *lie in* $D(c_1, R_1 + \rho_1)$.

II. *Setting* $L = m^m(|c_2 - c_1| + R_1 + R_2)^m / n^n(m-n)^{m-n}$, *the equation*

$$D(x) \equiv |\lambda| x^n - (x + |c_2 - c_1| + R_1 + R_2)^m = 0 \tag{10}$$

has two positive roots ρ_2, ρ_3 ($\rho_2 \leqq \rho_3$), *provided* $|\lambda| \geqq L$. *At least n zeros of* $F(z, \lambda)$ *lie in* $D(c_2, R_2 + \rho_2)$.

REMARK. Again, as in Theorem 3, the above statements can be weakened but made more practical by replacing the disks $D(c_1, R_1 + \rho_1)$ and $D(c_2, R_2 + \rho_2)$ by the disks $D(c_1, R_1 + y_1)$ and $D(c_2, R_2 + y_2)$, where y_1 and y_2 satisfy the inequalities $C(y_1) > 0$ and $D(y_2) > 0$.

PROOF. It follows from Theorem 2 that the zeros of $F(z, \lambda)$ are identical with the zeros of $G(z, \lambda) \equiv (z - \alpha_1)^m + \lambda(z - \alpha_2)^n$, where α_1 and α_2 are fixed, though undetermined, points in the disks $D(\alpha_1, R_1)$ and $D(\alpha_2, R_2)$, respectively. If we compare the equations (3) and (9) we see that since $|\alpha_2 - \alpha_1| \leqq |c_2 - c_1| + R_1 + R_2$, we have $r_1 \leqq \rho_1$. Similarly, comparing (4) with (10), we have $r_2 \leqq \rho_2 \leqq \rho_3 \leqq r_3$. To conclude the proof it is sufficient to refer to Theorem 3 and to note that $D(\alpha_1, r_1) \subset D(c_1, R_1 + \rho_1)$ and $D(\alpha_2, r_2) \subset D(c_2, R_2 + \rho_2)$ regardless of the exact location of α_1 and α_2.

4. Linear combinations of several polynomials. As mentioned earlier, the case $m = n$ was studied by Walsh [5] who obtained the following result:

THEOREM 5. *With the notations of Theorem 4, if* $m=n$, *then the zeros of* $F(z,\lambda)\equiv f_n(z)+\lambda g_n(z)$, $\lambda\neq 1$, *are in the union of the disks*

$$D\left(\frac{c_1-(-\lambda)^{1/n}c_2}{1-(-\lambda)^{1/n}},\ \frac{R_1+|\lambda|^{1/n}R_2}{|1-(-\lambda)^{1/n}|}\right),$$

where $(-\lambda)^{1/n}$ *takes all possible values. Any one of these* n *disks which is external to all others contains precisely one zero.*

The methods used in §3 yield the following result which is also a corollary to Theorem 5 inasmuch as the disk in the corollary contains the n disks in the theorem:

COROLLARY. *With the notations of Theorem 4, if* $m=n$ *and if* $|\lambda|<1$, *the zeros of* $F(z,\lambda)\equiv f_n(z)+\lambda g_n(z)$ *lie in the disk* $D(c_1, R_1+\rho_1)$, *where* ρ_1 *is the unique positive root of the equation*

$$x^n-|\lambda|(x+|c_2-c_1|+R_1+R_2)^n=0, \tag{11}$$

that is,

$$\rho_1=\frac{|\lambda|^{1/n}(|c_2-c_1|+R_1+R_2)}{1-|\lambda|^{1/n}}. \tag{12}$$

Before turning to linear combinations of several polynomials of the same degree n, we establish a lemma which is of some interest in its own right:

LEMMA. *Given the* k $(\geqq 2)$ *nonzero arbitrary complex numbers* $\lambda_1,\cdots,\lambda_k$ *which are subject to the conditions*:

I. $|\lambda_i|\neq|\lambda_j|$ *whenever* $i\neq j$.

II. *The sum of any number among the* λ*'s is never* 0, *we can always relabel the* λ*'s so that*

$$\begin{gathered}|\lambda_k|<|\lambda_{k-1}+\cdots+\lambda_1|,\ |\lambda_{k-1}|<|\lambda_{k-2}+\cdots+\lambda_1|,\cdots,\\ |\lambda_3|<|\lambda_2+\lambda_1|,\ |\lambda_2|<|\lambda_1|.\end{gathered} \tag{13}$$

PROOF. The case $k=2$ is trivial. Suppose then that $k\geqq 3$ and that the lemma is valid for $2,\cdots,k-1$ complex numbers satisfying Conditions I and II. To prove our lemma by induction, it suffices to show that among the k numbers there is one, say λ_k, which satisfies $|\lambda_k|<|\lambda_{k-1}+\cdots+\lambda_1|$. (The remaining inequalities in (13) will hold true by the assumption in the induction.) Indeed, otherwise, if $|\lambda_i|\geqq|S-\lambda_i|$ for $1\leqq i\leqq k$, where $S=\sum_1^k\lambda_i$, squaring and adding the k inequalities we would have $\sum_1^k|\lambda_i|^2\geqq\sum_1^k|S-\lambda_i|^2$ or $0\geqq k|S|^2-\sum_1^k(\lambda_i\bar{S}+\bar{\lambda}_iS)=(k-2)|S|^2$, which is false.

THEOREM 6. *Let* $f_i(z) \equiv z^n + a_{n-1}^{(i)} z^{n-1} + \cdots + a_0^{(i)}$, $1 \leqq i \leqq k$, *be* $k \geqq 2$ *polynomials of the same degree* n *whose zeros lie in the disks* $D(c_i, R_i)$, *respectively. Let* $F(z; \lambda_1, \cdots, \lambda_k) \equiv \lambda_1 f_1(z) + \cdots + \lambda_k f_k(z)$, *where the* λ_i*'s are complex numbers satisfying Conditions* I *and* II *in the lemma and thus will be assumed to satisfy also the inequalities* (13). *Then the* n *zeros of* $F(z; \lambda_1, \cdots, \lambda_k)$ *are in the disk* $D(c_1, R_1 + \sum_1^{k-1} \rho_i)$ *where the* ρ_i*'s are determined recursively as follows*: $\rho_0 = 0$, *and for* $1 \leqq i \leqq k-1$, ρ_i *is the unique positive root of the equation*

$$(14) \quad x^n - \left| \lambda_{i+1} \Big/ \sum_1^i \lambda_j \right| \left(x + |c_{i+1} - c_1| + R_1 + R_{i+1} + \sum_0^{i-1} \rho_j \right)^n = 0.$$

PROOF. The case $k=2$ follows from the corollary to Theorem 5. We proceed with the proof by induction. Suppose our theorem holds true for a linear combination of $k-1$ polynomials. Then the zeros of $F(z; \lambda_1, \cdots, \lambda_{k-1})$ are in $D(c_1, R_1 + \sum_1^{k-2} \rho_i)$ where the ρ_i's are determined again by (14). But

$$\begin{aligned} F(z; \lambda_1, \cdots, \lambda_{k-1}, \lambda_k) &= F(z; \lambda_1, \cdots, \lambda_{k-1}) + \lambda_k f_k(z) \\ &= (\lambda_1 + \cdots + \lambda_{k-1}) g_{k-1}(z) + \lambda_k f_k(z), \end{aligned}$$

where $g_{k-1}(z) \equiv z^n + \cdots$ is a polynomial of degree n. Applying the corollary to the last linear combination of two polynomials of equal degree gives us $D(c_1, R_1 + \sum_1^{k-1} \rho_i)$ as a disk containing the zeros of $F(z; \lambda_1, \cdots, \lambda_k)$.

Theorem 6 permits us, except in some special cases, to replace by a single polynomial any group of polynomials of the same degree that appears in a linear combination of several polynomials. We shall therefore assume in our final theorem that the polynomials under consideration have distinct degrees:

THEOREM 7. *Let* $f_i(z) \equiv z^{n_i} + a_{n_i-1}^{(i)} z^{n_i-1} + \cdots + a_0^{(i)}$, $1 \leqq i \leqq k$, $k \geqq 2$, *be* k *polynomials whose zeros lie in the disks* $D(c_i, R_i)$, *respectively. Suppose* $n_1 < n_2 < \cdots < n_k$. *Let* $F(z; \lambda_1, \cdots, \lambda_k) \equiv \sum_1^k \lambda_i f_i(z)$, *where the* λ_i,*s are arbitrary nonzero complex numbers. Then the* n_k *zeros of* $F(z; \lambda_1, \cdots, \lambda_k)$ *lie in the disk* $D(c_k, R_k + \rho_k)$, *where* ρ_k *is determined recursively as follows*: $\rho_1 = 0$, *and for* $2 \leqq i \leqq k$, ρ_i *is the unique positive root of the equation*

$$(15) \quad A_i(x) \equiv x^{n_i} - |\lambda_{i-1}/\lambda_i| (x + |c_i - c_{i-1}| + R_i + R_{i-1} + \rho_{i-1}) x^{n_i - 1} = 0.$$

REMARK. A remark similar to the ones made following Theorems 3 and 4 can be made here: in the statement of Theorem 7 the roots ρ_i could be replaced by numbers $y_i > 0$ for which $A_i(y_i) > 0$.

PROOF. If we define the polynomials $h_i(z)$ recursively as follows: $h_1(z) \equiv f_1(z)$, and for $2 \leqq i \leqq k$, $h_i(z) \equiv f_i(z) + (\lambda_{i-1}/\lambda_i)h_{i-1}(z)$, we see that $F(z; \lambda_1, \cdots, \lambda_k) \equiv \lambda_k h_k(z)$. Thus $h_i(z)$ is a linear combination of two polynomials to which Theorem 4 is applicable and the truth of Theorem 6 follows by induction.

REFERENCES

1. M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Math. Surveys No. 3, Amer. Math. Soc., Providence, R. I., 1949.

2. A. Ostrowski, *Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent*, Acta Math. **72** (1940), 99–257.

3. Z. Rubinstein, *Analytic methods in the study of zeros of polynomials*, Pacific J. Math. **13** (1963), 237–249.

4. W. Specht, *Algebraische Gleichungen mit reelen oder komplexen Koeffizienten*, Enzyklop. d. math. Wissen., 2nd ed., I 1, 3, II, Stuttgart, 1958.

5. J. L. Walsh, *On the location of roots of certain types of polynomials*, Trans. Amer. Math. Soc. **24** (1922), 163–180.

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ZEROS OF EXPONENTIAL SUMS[1]

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1. Introduction. This paper deals with the distribution in the complex plane of the zeros of an exponential sum of the form

$$f(z) = \sum_{j=1}^{n} A_j z^{m_j}[1 + \epsilon_j(z)]e^{\omega_j z}, \tag{1}$$

where $n > 1$; the A_j and the ω_j are complex numbers such that $A_j \neq 0$ and the ω_j are distinct; the m_j are non-negative integers; the functions ϵ_j are analytic for $|z| \geqq r_0 \geqq 0$ with $\lim_{z\to\infty} \epsilon_j(z) = 0$. A function of the form

$$\sum_{j=1}^{n} P_j(z)e^{\omega_j z}, \tag{2}$$

Received by the editors September 9, 1963.

[1] This work was supported by a grant from the National Science Foundation.