Proof. If we define the polynomials $h_i(z)$ recursively as follows: $h_1(z)\equiv f_1(z)$, and for $2\leq i\leq k$, $h_i(z)\equiv f_i(z)+(\lambda_{i-1}/\lambda_i)h_{i-1}(z)$, we see that $F(z;\lambda_1,\cdots,\lambda_k)\equiv\lambda_k h_k(z)$. Thus $h_i(z)$ is a linear combination of two polynomials to which Theorem 4 is applicable and the truth of Theorem 6 follows by induction.

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ZEROS OF EXPONENTIAL SUMS[1]

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1. Introduction. This paper deals with the distribution in the complex plane of the zeros of an exponential sum of the form

$$f(z) = \sum_{j=1}^{n} A_j z^{m_j}[1+\epsilon_j(z)]e^{\omega_j z}, \tag{1}$$

where $n>1$; the A_j and the ω_j are complex numbers such that $A_j\neq 0$ and the ω_j are distinct; the m_j are non-negative integers; the functions ϵ_j are analytic for $|z|\geq r_0\geq 0$ with $\lim_{z\to\infty}\epsilon_j(z)=0$. A function of the form

$$\sum_{j=1}^{n} P_j(z)e^{\omega_j z}, \tag{2}$$

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where the P_j are polynomials, is an important special case of (1).

When the P_j in (2) are constants and the ω_j are real, the zeros of (2) are confined to a vertical strip in the plane. C. E. Wilder [8] found an estimate of the number of zeros in a rectangle with vertical sides on the boundary of such a strip. A strengthening by Langer [3] of Wilder's result showed that the number of zeros in such a rectangle of height s with a zero-free boundary differs from max $|\omega_i - \omega_j| s/(2\pi)$ by at most $n-1$. Allowing the ω_j to be complex, Tamarkin [7] found similar results for rectangles in strips in the direction of the exterior normals to the sides of the polygonal convex hull Q of the set $\{\bar{\omega}_j\}_{j=1}^n$. These results together with the results of Pólya [4] and Schwengeler [6], which located the zeros of (2) of large modulus in strips that are logarithmic to the normals, were combined by Langer to obtain similar asymptotic results for zeros of (1) in sets that are asymptotically rectangular.

Our purpose here is to obtain results in which this asymptotic similarity is made explicit. Some results in this direction were obtained by the author in [1]. However, those results were limited in generality (in particular, regarding the dependence of parameters), having been designed for a limited application. An application of the more general results obtained here will be made elsewhere [2] to establish Pólya's remark [5, p. 549] to the effect that the number of zeros of (2) of modulus at most r equals $Lr/(2\pi)+O(1)$, where L is the perimeter of Q. This is a special case of a similar result for (1).

We will rely rather heavily on the notation and results in the early part of [1]. The notation will be described briefly in §2. In §3 a very slight strengthening of Langer's proof of Wilder's theorem will yield a best possible result. This will then be used to prove the main result given in Theorem 2.

2. **Preliminaries.** We will use the generic symbol $\epsilon(z)$ to denote a function which is analytic for large $|z|$ in a region R and which has the property that $\epsilon(z)\to 0$ as $z\to\infty$ in R. For convenience, by the zeros of f we will mean the zeros of f of modulus greater than r_0.

In [1] the indices on the ω_j of (1) are labeled so that $\bar{\omega}_k$, for $k=1, \cdots, \sigma$, are the vertices of Q given in a specific counterclockwise order. L_k is the line segment $[\bar{\omega}_k, \bar{\omega}_{k+1}]$, and ϕ_k is the argument of $\bar{\omega}_k-\bar{\omega}_{k+1}$ in $[-\pi/2, 3\pi/2)$. Let $e_k=\exp(i\phi_k)$. Certain $\bar{\omega}_p$ on L_k are assigned doubly indexed subscripts as follows: Consider the convex hull of $\bar{\omega}_k$, $\bar{\omega}_{k+1}$ and the $\tau_p=\bar{\omega}_p+im_pe_k$ for which $\bar{\omega}_p$ is on L_k; assign subscripts $j=1, \cdots, \sigma_k$ to ω_{kj} so that $\omega_{k1}=\omega_k$, $\omega_{k\sigma_k}=\omega_{k+1}$, and τ_{kj} are vertices of this hull proceeding in a counterclockwise direction from $\bar{\omega}_k+im_ke_k$ to $\bar{\omega}_{k+1}+im_{k+1}e_k$. For $j=1, \cdots, \sigma_k-1$, $L_{kj}=[\tau_{kj}, \tau_{kj+1}]$,

μ_{kj} is the quotient of $m_{kj}-m_{kj+1}$ by $(\omega_{kj}-\omega_{kj+1})e_k$ and is real, and n_{kj} is the number of τ_p on L_{kj}.

For $k=1, \cdots, \sigma, j=1, \cdots, \sigma_k-1$, and $H>0$, the set $V_{kj}(H)$ is the set

$$\{z; \mathscr{I}(z/e_k) \geqq 0, \mid \mathscr{R}(z/e_k) + \mu_{kj} \log |z| \mid \leqq H\}.$$

Setting $z'=z/e_k$, it is easy to show that for a fixed H the subsets of the $V_{kj}(H)$ composed of z of large modulus are individually connected and mutually disjoint with respect to pairs (k, j). The boundary curves of $V_{kj}(H)$ are logarithmic to the exterior (to Q) normal to L_k through the origin. For each k and for each θ in $(0, \pi/2)$, $T_k(\theta)$ is a closed sector with vertex at the origin of opening 2θ about the outward normal to L_k through the origin. For fixed θ and H, the subsets of $V_{kj}(H)$ composed of points of sufficiently large modulus are in $T_k(\theta)$.

For the same k and j and each triple of reals (α, s, H), where $s>0$ and $H>0$, the set $R_{kj}(\alpha, s, H)$ is given by

$$\{z; \mathscr{I}(z/e_k) + \mu_{kj} \arg z \in [\alpha, \alpha+s], \mid \mathscr{R}(z/e_k) + \mu_{kj} \log |z| \mid \leqq H\},$$

where $\arg z \in (\phi_k, \phi_k+\pi)$. For fixed θ and H, $R_{kj}(\alpha, s, H)$ is in $V_{kj}(H) \cap T_k(\theta)$ for all $s>0$ if α is sufficiently large. If α is large, the set approximates a rectangle of dimensions s by $2H$. $N_f(R_{kj}(\alpha, s, H))$ denotes the number of zeros of f in closed $R_{kj}(\alpha, s, H)$, the zeros being counted according to their multiplicities.

For triples of reals (α, s, H) with $s>0$ and $H>0$, $R(\alpha, s, H)$, without subscripts, denotes the rectangle of points $z=x+iy$ with $|x| \leqq H$ and y in $[\alpha, \alpha+s]$. $N_g(R(\alpha, s, H))$ denotes the number of zeros of the function g in that rectangle.

3. Distribution of zeros.

We first give Wilder's theorem, adapting Langer's proof so as to include all rectangles. In particular, the function in question may have zeros on the boundary of a rectangle.

THEOREM 1. *Let* $g(z)=\sum_{j=1}^{n} A_j \exp(\omega_j z)$, *where* $z=x+iy$, $A_j \neq 0$, $\omega_1<\omega_2< \cdots <\omega_n$. *Then there exists* $K>0$ *such that* (a) *each zero of* g *is in* $|x|<K$, *and* (b) *for each pair of reals* (α, s) *with* $s>0$,

$$| N_g(R(\alpha, s, K)) - s(\omega_n - \omega_1)/(2\pi) | \leqq n-1.$$

PROOF. Let $h(z)=g(z)\exp(-\omega_1 z)/A_1$. Then h and g have the same zeros and $h(z)=1+\sum_{j=2}^{n} B_j \exp(\gamma_j z)$, where $\gamma_j=\omega_j-\omega_1$, $B_j=A_j/A_1$ and $0=\gamma_1< \cdots <\gamma_n$. For $x>0$, $h(z)=[1+\beta_1(z)]B_n \exp(\gamma_n z)$, where $\beta_1(z)\to 0$ as $x\to\infty$, while for $x<0$, $h(z)=1+\beta_2(z)$, where $\beta_2(z)\to 0$ as $x\to-\infty$. If K is chosen so that $|\beta_i(z)|<1/2$ when $|x|\geqq K$, conclusion (a) follows.

To establish (b) it suffices to show that for each $\epsilon > 0$

$$\left| N_h(R(\alpha, s, K)) - s\gamma_n/(2\pi) \right| < n - 1 + \epsilon.$$

Given $\epsilon > 0$, choose $H_\epsilon \geqq K$ so that $|\arg[1+\beta_i(z)]| < \pi\epsilon/4$ respectively on the lines $x = (-1)^{i+1}H_\epsilon$, $i = 1, 2$. For each $s > 0$, choose $\delta = \delta(\epsilon, s) > 0$ so that $\delta < \pi\epsilon/(2\gamma_n)$, $\delta < s/2$, and so that h has no zeros on the boundaries of

$$R_1 = R(\alpha - \delta, s + 2\delta, H_\epsilon)$$

and

$$R_2 = R(\alpha + \delta, s - 2\delta, H_\epsilon).$$

This is possible since the zeros of h are isolated. Consider changes in arg $h(z)$ as z traverses R_1 in the positive direction. On the right boundary the change is at most $(s+2\delta)\gamma_n + \pi\epsilon/2$, while on the left it is at most $\pi\epsilon/2$. On a line where y is constant, $\mathscr{I}(h(z))$ has the form $\sum_{j=2}^{n} E_j \exp(\gamma_j x)$ with E_j real. By induction, such a function, if not identically zero, can have at most $n-2$ real zeros. Hence on a horizontal side of R_1, either $\mathscr{I}(h(z)) \equiv 0$ and the change in arg $h(z)$ is zero since h has no zeros there, or the absolute value of the change is less than $(n-1)\pi$. Therefore the total change is bounded above by $(s+2\delta)\gamma_n + \pi\epsilon + 2\pi(n-1)$. By our choice of δ, it follows that $N_h(R(\alpha, s, K)) \leqq N_h(R_1) < n - 1 + \epsilon + s\gamma_n/(2\pi)$. Similarly, $N_h(R(\alpha, s, K)) \geqq N_h(R_2) > -n + 1 - \epsilon + s\gamma_n/(2\pi)$. The conclusion then follows.

We note that the conclusions of this theorem hold for each $H \geqq K$. That this is the best possible result involving all rectangles, i.e., all α and s, can be verified with the function $1 + e^z$.

THEOREM 2. *Given f as in* (1), *then there exists $K > 0$ such that* (a) *all but a finite number of zeros of f of modulus greater than r_0 are in* $\bigcup_{k,j} V_{kj}(K)$, *and* (b) *for each pair of positive reals ϵ and s_0, there exists an $\alpha_0 = \alpha_0(\epsilon, s_0)$ such that whenever $\alpha \geqq \alpha_0$ and $s \geqq s_0$,*

$$\left| N_f(R_{kj}(\alpha, s, K)) - s\left|\omega_{kj+1} - \omega_{kj}\right|/(2\pi) \right| < n_{kj} - 1 + \epsilon.$$

Before proving the theorem, we remark that the proofs of the theorems in [1, pp. 17, 21] that correspond to the theorems here, warrant only weak inequalities in their conclusions. In the second theorem there (in which the last sentence should be deleted), the magnitude of s depends on h as well as H. This is a deficiency for some applications and is not found in the more general theorem given here.

PROOF. Let $g_{kj}(z) = \sum A_{kp} \exp[(\omega_{kp} - \omega_{kj})e_k z]$ where the sum is taken over p for which $\tau_{kp} \in L_{kj}$. Choose $K > 0$ sufficiently large so that the conclusion of Lemma 13 of [1] holds when $H = K$ (e.g., by insuring that $\sum_{p\neq j} |A_{kp}/A_{kj}| \exp[-K|\omega_{kp}-\omega_{kj}|]<1/4)$ and so that Theorem 1 holds for K with each g_{kj}. Part (a) then follows from Theorem 1 of [1] because of the first restriction on K.

Choose $r_1>0$ so that $r_1 \geqq r_0$, all zeros of f of modulus greater than r_1 are in $\bigcup_{k,j} V_{kj}(K)$, and so that the sets $V_{kj}(K+1) \cap \{z;\ |z|>r_1\}$ are mutually disjoint with respect to pairs (k, j). Let θ be fixed in $(0, \pi/2)$, yielding sectors $T_k(\theta)$. Choose $\alpha_1>0$ sufficiently large so that if $\alpha \geqq \alpha_1$, then for all $s>0$, $R_{kj}(\alpha, s, K+1)$ is contained in

$$\Omega = V_{kj}(K+1) \cap T_k(\theta) \cap \{z;\ |z| > r_1\}.$$

By Lemma 21 of [1], for all z in Ω, $f(z)$ is given by

$$(3) \quad z^{m_{kj}} e^{\omega_{kj} z} \sum A_{kp}[1+\epsilon(z)] \exp[e_k(\omega_{kp}-\omega_{kj})(z/e_k+\mu_{kj}\log z)],$$

where the sum is taken over all p for which $\tau_{kp} \in L_{kj}$. The form of the $\epsilon(z)$ depends on θ, and $\epsilon(z) \to 0$ as $z \to \infty$ in Ω. The transformation $w=u+iv=z/e_k+\mu_{kj}\log z$, where $\mathcal{I}(\log z) \in (\phi_k, \phi_k+\pi)$ maps Ω into the strip $|u| \leqq K+1$ with sets $R_{kj}(\alpha, s, K+\delta)$ mapping conformally onto rectangles $R(\alpha, s, K+\delta)$ in the w-plane when $\alpha \geqq \alpha_1$, $s>0$, and $0 \leqq \delta \leqq 1$.

Define f_{kj} and h_{kj} by $f(z)=f_{kj}(z) \exp[m_{kj} \log z+\omega_{kj} z]$ and $h_{kj}(w) =f_{kj}(z)$, where $w=z/e_k+\mu_{kj} \log z$. It follows easily from (3) that $f_{kj}(z)=g_{kj}(w)+\epsilon(w)$ for z in Ω, where $\epsilon(w) \to 0$ as $w \to \infty$ in $|u| \leqq K+1$. Hence if $\alpha \geqq \alpha_1$, $s>0$, and $0 \leqq \delta \leqq 1$, then $N_f(R_{kj}(\alpha, s, K+\delta)) =N_{f_{kj}}(R_{kj}(\alpha, s, K+\delta))=N_{h_{kj}}(R(\alpha, s, K+\delta))$.

For given positive ϵ and s_0, choose $\delta=\delta(\epsilon, s_0)$ so that

$$\delta < \pi\epsilon/|\omega_{kj+1}-\omega_{kj}|, \ \delta < s_0/2, \text{ and } \delta < 1.$$

Noting that the exponent coefficients in g_{kj} are real, it follows from Theorem 1 that there is a bound on the number of zeros of g_{kj} in $R(\alpha+s, \delta, K+\delta)$ and $R(\alpha-\delta, \delta, K+\delta)$ which is independent of α and s but dependent on δ and hence on ϵ and s_0. As a result, in each of these rectangles one may pass a horizontal line that is uniformly (in α and s) bounded from the zeros of g_{kj}. For $s>0$ and $\alpha \geqq \alpha_1+s_0/2$, let $R_{\alpha s}$ be a rectangle formed by such lines and the lines $u=\pm(K+\delta)$. Then $R_{\alpha s} \supset R(\alpha, s, K)$, and for w on the boundary of $R_{\alpha s}$, $h_{kj}(w)=g_{kj}(w)[1+\epsilon(w)/g_{kj}(w)]$. When w is on the boundary of the $R_{\alpha s}$, w is uniformly bounded from the zeros of g_{kj}. By Theorem 3 of [1] it follows that $|g_{kj}(w)|$ is uniformly bounded from zero for such w. Choose α_0 sufficiently large so that $\alpha_0 \geqq \alpha_1 +s_0/2$, and if $\alpha \geqq \alpha_0$ and w is on the boundary of $R_{\alpha s}$, then $|\epsilon(w)/g_{kj}(w)| <1$. Then for $\alpha \geqq \alpha_0$ and $s>0$, h_{kj} and g_{kj} have the same number of zeros in $R_{\alpha s}$. Using the second condition in the choice of K

and the fact that $\delta < \pi\epsilon / |\omega_{kj+1} - \omega_{kj}|$, for $\alpha \geqq \alpha_0$ and $s > 0$, it follows that $N_f(R_{kj}(\alpha, s, K)) = N_{h_{kj}}(R(\alpha, s, K)) \leqq N_{h_{kj}}(R_{\alpha s}) = N_{g_{kj}}(R_{\alpha s}) \leqq s|\omega_{kj+1} - \omega_{kj}|/(2\pi) + n_{kj} - 1 + \epsilon$.

Since $\delta < s_0/2$, when $s \geqq s_0$ one may construct rectangles with vertical sides on $u = \pm(K+\delta)$ and zero-free horizontal sides inside and within δ of those of $R(\alpha, s, K)$ on which w is uniformly bounded from the zeros of g_{kj}. By our choice of r_1 and α_1, $N_f(R_{kj}(\alpha, s, K)) = N_f(R_{kj}(\alpha, s, K+\delta))$ when $\alpha \geqq \alpha_1$ and $s > 0$. An argument like the above (in which α_0 may need to be increased) gives as a lower bound for $N_f(R_{kj}(\alpha, s, K))$ the quantity $s|\omega_{kj+1} - \omega_{kj}|/(2\pi) - n_{kj} + 1 - \epsilon$ if $s \geqq s_0$. Together, the two bounds establish (b).

If ϵ is chosen to be one, then for a fixed $s > 0$ one can use the bound n_{kj} in (b) for all large α. If with $\epsilon = 1/2$ one considers only s for which $s|\omega_{kj+1} - \omega_{kj}|/(2\pi)$ is a positive integer (such s are automatically bounded from zero), then $n_{kj} - 1$ serves as a weak bound in (b) for all large α.

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