A NEW PROOF OF A CONJECTURE OF SCHILD

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1. **Introduction.** In a paper, published in this Journal [3], one of the authors has introduced and discussed the class of functions S_p, having $|z|=1$ as radius of schlichtness and being of the form $f_p(z) = z - \sum_{n=2}^{N} a_n z^n$, with the a_n real and non-negative for $n=2, 3, \cdots, N$, $N \geqq 2$. All functions of this class map the unit circle into starlike regions [3, Theorem 3].

Let d^* be the shortest distance from $w=0$ to $w=f_p(e^{i\theta})$, $0 \leqq \theta < 2\pi$, and d_0 the shortest distance from $w=0$ to $w=f_p(r_0 e^{i\theta})$, $0 \leqq \theta < 2\pi$, where r_0 is the radius of convexity of $w=f_p(z)$. Among other things it was proved [3, Theorem 7] that for all $f_p(z) \in S_p$ we have $d_0/d^* \geqq 2/3$. It was conjectured there that for $f_p(z) \in S_p$ we actually have $d_0/d^* \geqq 3/4$, attained by the function $f_p(z) = z - z^2/2 \in S_p$.

The class of functions S_p was discussed further and extended by Z. Lewandowski [1] and the truth of the conjecture $d_0/d^* \geqq 3/4$ for all $f_p(z) \in S_p$ was demonstrated by him in a second paper [2].

It is the aim of this short note to give an elementary and simple proof of the conjecture: $d_0/d^* \geqq 3/4$ for all $f_p(z) \in S_p$.

2. **Proof of the conjecture that $d_0/d^* \geqq 3/4$ for all $f_p(z) \in S_p$.** The map of $|z| = r$, $0 < r \leqq 1$, by any $f_p(z) \in S_p$ will have its closest point from the origin on the positive real axis for $z = r$, since

$$|f_p(z)| = \left| z - \sum_{n=2}^{N} a_n z^n \right| \geqq |z| - \sum_{n=2}^{N} a_n |z|^n = r - \sum_{n=2}^{N} a_n r^n = f_p(r).$$

We must show, therefore, that

$$d_0/d^* = \left\{ r_0 - \sum_{n=2}^{N} a_n r_0^n \right\} \Big/ \left\{ 1 - \sum_{n=2}^{N} a_n \right\} \geqq 3/4,$$

where r_0 is the radius of convexity of $f_p(z)$.

$$\begin{aligned} d_0/d^* - 3/4 &= \left\{ r_0 - \sum_{n=2}^{N} a_n r_0^n \right\} \Big/ \left\{ 1 - \sum_{n=2}^{N} a_n \right\} - 3/4 \\ &= \left\{ r_0 - 3/4 + \sum_{n=2}^{N} a_n [3/4 - r_0^n] \right\} \Big/ \left\{ 1 - \sum_{n=2}^{N} a_n \right\}. \end{aligned}$$

Since $\{1 - \sum_{n=2}^{N} a_n\} > 0$ [3, Theorem 1], it is sufficient to show that $y = (r_0 - 3/4) + \sum_{n=2}^{N} a_n (3/4 - r_0^n) \geqq 0$.

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We recall [3, Theorem 5] that r_0 is the least positive root of $\sum_{n=2}^N n^2a_nr^{n-1}=1$ and, therefore, $\sum_{n=2}^N n^2a_nr_0^{n-1}=1$. The expression for y can now be written in the form:

$$y = (r_0 - 3/4)\sum_{n=2}^N n^2 a_n r_0^{n-1} + \sum_{n=2}^N a_n(3/4 - r_0^n)$$

$$= \sum_{n=2}^N a_n\{n^2 r_0^n - (3/4)n^2 r_0^{n-1} + 3/4 - r_0^n\}.$$

The proof will now be completed by showing that

$$z(n) = \{n^2 r_0^n - (3/4)n^2 r_0^{n-1} + 3/4 - r_0^n\} \geqq 0 \quad \text{for } n = 2, 3, \cdots, N.$$

Clearly, $z(2)=3r_0^2-3r_0+3/4=3(r_0-1/2)^2\geqq 0$. For $n\geqq 2$, we consider

$$g(n) = z(n+1) - z(n)$$

$$= r_0^{n-1}\{(n^2 + 2n)r_0^2 - (7n^2/4 + 3n/2 - 1/4)r_0 + 3n^2/4\}.$$

It was shown [4, Lemma 3.4] that $d_0/d^*>r_0$, for all functions $w=z+\sum_{n=2}^N a_nz^n$, regular, schlicht and starlike in the unit circle. This result will, therefore, also hold for $f_p(z)\in S_p$, and since, for this class of functions, $r_0\geqq 1/2$ [3, Theorem 5], it is sufficient to prove the conjecture for $1/2\leqq r_0<3/4$. It is, therefore, convenient to set $r_0=3/4-x$, where $0<x\leqq 1/4$, in the expression for $g(n)$. For any particular n, the coefficient of r_0^{n-1} in $g(n)$ becomes $h(x)=(n^2+2n)x^2+(1/4)(n^2-6n-1)x+3/16$. It is clear that $h(x)>0$ for $n\geqq 6$. Also, the discriminant of $h(x)$ is $\Delta=(1/16)(n^4-12n^3+22n^2-12n+1)=(1/16)(n-1)^2(n^2-10n+1)$. Obviously, $\Delta<0$ for $n<10$, and since $h(0)>0$, $h(x)>0$ for $n=2, 3, 4, 5$ also. Therefore, $z(n+1)-z(n)>0$ for $n=2, 3, 4, \cdots$ and since $z(2)\geqq 0$, the conjecture is proved.

References

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