A UNIQUENESS THEOREM FOR ENTIRE FUNCTIONS

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Given a sequence \( \{ \mathcal{L}_n \} \) of linear functionals defined on a linear space \( C \) of functions, the uniqueness problem is to find a subspace \( C_1 \subseteq C \) such that \( f \) in \( C_1 \) is uniquely determined by the sequence of numbers \( \{ \mathcal{L}_n(f) \} \); that is, \( g \in C_1 \) and \( \mathcal{L}_n(g) = 0 \); \( n = 0, 1, 2, \ldots \) implies \( g = 0 \).

We shall use the following notation. Let \( K \) denote the class of all entire functions of exponential type. If \( \Omega \) is a simply-connected domain in the complex plane, let \( K[\Omega] \) denote the class of all \( f \) in \( K \) such that the Borel transform of \( f \), which we shall denote by \( F \), is regular on \( \Omega' \), the complement of \( \Omega \). (If \( f(z) = \sum a_n z^n/n! \), then \( F(\zeta) = \sum a_n \zeta^{-n-1} \).) We shall deal with sequences \( \{ \mathcal{L}_n \} \) of linear functionals defined on a class \( K[\Omega] \) by

\[
\mathcal{L}_n(f) = \frac{1}{2\pi i} \int_{\Gamma} [W(\zeta)]^n F(\zeta) \, d\zeta
\]

for some function \( W \) regular on \( \Omega \), where \( \Gamma \) is a simple contour contained in \( \Omega \) and enclosing all singularities of \( F \). This class of functionals has been studied by Gelfond [5], Buck [3], and the author [4]. Examples of functionals having a representation (1) are \( \mathcal{L}_n(f) = f(n) \), \( \mathcal{L}_n(f) = \Delta^m f(0) \), and \( \mathcal{L}_n(f) = f^{(n)}(n) \) for which the functions \( W(\zeta) \) are \( e^\zeta \), \( e^{\zeta - 1} \), and \( \zeta e^\zeta \), respectively.

Previous uniqueness theorems which apply to the class of sequences of functionals having a representation (1) have been obtained by finding a class \( K[\Omega] \) of functions \( f \) representable by an interpolation series having the \( \mathcal{L}_n(f) \) as coefficients. If for all \( f \) in \( K[\Omega] \) this series converges to \( f \) in some region or is summable by some totally regular method of summation, then \( K[\Omega] \) is a uniqueness class for \( \{ \mathcal{L}_n \} \). The best such sufficient condition was obtained by Buck [3] using Mittag-Leffler summability. It states that if \( \Omega \) contains the origin, \( W(0) = 0 \), \( W \) is univalent on \( \Omega \) and maps \( \Omega \) onto a set which is star-shaped with respect to the origin, then \( K[\Omega] \) is a uniqueness class for \( \{ \mathcal{L}_n \} \). Gelfond [5] showed that the condition that \( W \) is univalent on \( \Omega \) is necessary for \( K[\Omega] \) to be a uniqueness class for \( \{ \mathcal{L}_n \} \).

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that this one condition is also sufficient. The proof is simple and does not use interpolation series. It is similar to a proof given by Boas and Buck [2, p. 26] of a theorem on sources of nontrivial representations of zero.

**Theorem.** Let $\Omega$ be a simply-connected domain and let $W$ be regular on $\Omega$. Let $L_n$ be defined on $K[\Omega]$ by (1) with this $W$. Then a necessary and sufficient condition for $K[\Omega]$ to be a uniqueness class for $\{L_n\}$ is that $W$ be univalent on $\Omega$.

**Proof.** We give Gelfond’s proof of necessity. If $W$ is not univalent on $\Omega$, then there exist $\zeta_1, \zeta_2$ in $\Omega$ such that $\zeta_1 \neq \zeta_2$, but $W(\zeta_1) = W(\zeta_2)$. Let $f(z) = e^{\zeta_1 z} - e^{\zeta_2 z}$. Then $L_n(f) = [W(\zeta_1)]^n - [W(\zeta_2)]^n = 0$; $n = 0, 1, 2, \ldots$, but $f \neq 0$.

To prove sufficiency, let $W$ be univalent on $\Omega$ and let $f$ in $K[\Omega]$ satisfy $L_n(f) = 0$; $n = 0, 1, 2, \ldots$, i.e.,

$$\int_{\Gamma} [W(\zeta)]^n F(\zeta) d\zeta = 0, \quad n = 0, 1, 2, \ldots .$$

Let $\Omega_w$ be the image of $\Omega$ and $\Gamma_w$ the image of $\Gamma$ under $w = W(\zeta)$. Since $W$ is regular and univalent on $\Omega$, $\Gamma_w$ is a simple contour enclosing the image of the region enclosed by $\Gamma$. Let $\zeta = Z(w)$ be the inverse of $W$ which maps $\Omega_w$ onto $\Omega$. Then $Z$ is regular and univalent on $\Omega_w$; so $Z'(w) \neq 0$ on $\Omega_w$. Since $\Gamma$ encloses all singularities of $F(\zeta)$, $\Gamma_w$ encloses all singularities of $F(Z(w))$; so, $F(Z(w))$ is analytic on $\Gamma_w$. Substituting $\zeta = Z(w)$ in the integrals, we obtain

$$\int_{\Gamma_w} w^n F(Z(w)) Z'(w) dw = 0, \quad n = 0, 1, 2, \ldots .$$

Multiplying the $n$th integral by $s^n/n!$ and summing, we obtain

$$\int_{\Gamma_w} e^{sz} F(Z(w)) Z'(w) dw = 0$$

for all $z$. Since $F(Z(w))Z'(w) = G(w)$ is analytic on $\Gamma_w$, this implies, by a lemma of Pólya [1, p. 110], that $G$ is analytic inside $\Gamma_w$. Then, since $Z'(w)$ is analytic and nonzero inside $\Gamma_w$, $F(Z(w)) = G(w)/Z'(w)$ is analytic inside $\Gamma_w$. Since $Z$ is analytic inside $\Gamma_w$, this implies $F$ is analytic inside $\Gamma$. But $F$ is also analytic outside and on $\Gamma$ and at $\infty$; so $F$ is a constant. Since $F(\infty) = 0$, $F = 0$, and this implies $f = 0$, which completes the proof.

A simple example of a specialization of the theorem is that if $\Omega$ is a simply-connected domain, then $K[\Omega]$ is a uniqueness class for
either \( \{f(n)\} \) or \( \{\Delta^n f(0)\} \) if and only if \( \xi \in \Omega \) implies \( \xi + 2n\pi i \notin \Omega \), \( n = 1, 2, \ldots \). From this, dropping the requirement that \( G \) must be open in the notation \( K[G] \), the class \( K[\Omega_i] \) is a uniqueness class for either of these sequences of functionals if \( \Omega_i \) is the strip \( \{x + iy\mid -\pi < y \leq \pi\} \). The class \( K[\Omega_i] \) is maximal in the sense that if \( \Omega_2 \) is any set which properly contains \( \Omega_i \), then \( K[\Omega_2] \) is not a uniqueness class. That \( K[\Omega_i] \) is a uniqueness class follows from the fact that if the Borel transform of a function \( f \) is analytic on the complement of \( \Omega_i \), then it is analytic on the complement of some open horizontal strip of width \( 2\pi \).

**Bibliography**


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