CERTAIN FINITE NONPROJECTIVE GEOMETRIES WITHOUT THE AXIOM OF PARALLELS

MASON HENDERSON

In this paper, "point," "line," and "incident upon," are undefined terms. The phrases, "point is on a line," "point is incident upon a line," "line is incident upon a point," "line is on a point," are all to be considered synonymous. We will say that a line l intersects a line k (at a point P) if and only if P is on both l and k. Two lines are parallel if and only if there is no point which is on both. The following axioms will be used:

Axiom I. If P and Q are distinct points, there is exactly one line on P and on Q.

Axiom II. If l is a line, there is at least one point not on l.

Axiom III. If l is a line and P is a point not on l, there are exactly m distinct lines on P ($m \geqq 2$) which are parallel to l.

Axiom IV. There is at least one line with exactly n points on it, $n \geqq 2$.

The entire set of points and lines whose existence is postulated by these axioms (for given m and n) will be called a geometry.

Other work done with an axiom system containing the Axiom III as here stated is not known to this investigator, but Szamkolowicz [3] has reported on an equivalent system, and similar systems have been studied [2], [4]. As they are here stated, the axioms may or may not be consistent, depending on the values assigned to m and n. For $n=2$, their consistency for any m is established by the existence of a model constructed by A. N. Milgram [4], and for $n=3$, $m=4$, their consistency is shown by a model described by Abraham Barshop [1]. This paper will demonstrate their inconsistency for $m=2$ and $n>2$.

Theorem I. *There are exactly $n+m$ lines on every point, and exactly n points on every line.*

The axioms specify the existence of one line, say l, with exactly n points on it, $n \geqq 2$. I will reserve the letter l for that line throughout the proof of this theorem. Then if P is any point not on l, P has n lines on it which are on the n points of l, and m lines on it parallel to l. Hence, there are exactly $n+m$ lines on any point P not on l.

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LEMMA 1. *Every line has at least one point on it.*

Consider line j. If j is l, the lemma is true. Suppose j is not l. There exists a point P not on l. If P is on j, the lemma is true. If not, then n distinct lines on P intersect j (P has $n+m$ lines on it, just m of which are parallel to j, so n of them are on j), necessarily at n distinct points. So in either case, j has at least one point on it.

LEMMA 2. *Every line has at least two points on it.*

If k is any line, then k may be l (and the lemma is true), or k may have one point on l, or be parallel to l. If k has only one point on l, then consider the existence of a point P not on l. If P is on k, the lemma is true. If P is not on k, then n of the $n+m$ lines on P are on k, by an argument similar to that used in Lemma 1; hence there are n points on k and the lemma is again true. Finally, k may be parallel to l. By Lemma 1 there is at least one point Q on k, and line QA exists, if A is a point of l. There is another point B on l, and m parallels to QA on B (B is obviously not on QA), each of which is distinct from l. If each of these m parallels is parallel to k also, then there are at least $m+1$ lines on B parallel to k (since l also is parallel to k), a contradiction. Hence, one of these m parallels to QA is on k at some point R. R cannot be Q, since R is on a parallel to QA, so k has at least two points on it in this case, and the lemma is proved.

LEMMA 3. *Given any two lines, j and k, there is a point not on j and not on k.*

There is certainly a point P not on j, and if it is not on k, the lemma holds; so it may be assumed that P is on k. On P there are at least two parallels to j, at least one of which is not k. Call it f. By Lemma 3, f has at least one point Q distinct from P. Further, Q is not on k (if so, $f=PQ=k$, a contradiction), and not on j, since it is on a parallel to j. The lemma follows.

From the last lemma, it is seen that if k is any line, there is a point P not on k, and not on l. Being a point not on l, P has $n+m$ lines on it, just m of which are parallel to k. Thus exactly n of the lines on P intersect k, necessarily at n distinct points; and there is no point on k which is not on some line on P; so there are exactly n points on any line k.

Finally, let P be any point on l. There exists Q, another point on l, and R, a point not on l, and the line QR. Then QR has exactly n points on it. P is not on QR. So the lines on P are the m parallels to QR and the n lines on P and on points of QR. Hence, if P is any point on l or not, P has exactly $n+m$ lines on it.

The number of points in a geometry may be established easily. There are $n+m$ lines on a given point, and all points of the geometry are on these lines. On each such line there are exactly $n-1$ points (excluding the given point). Thus there are $(n+m)(n-1)+1$ points in a geometry. In what follows, it will be assumed that $m=2$ and $n>2$. Accordingly, the number of points in a geometry is n^2+n-1.

There exists a line x, which is parallel to a given line y. At each of the points of x, there are two lines parallel to y (one of which is x, of course). The one of each pair which is not x will be called a bar, and the set of lines composed of x, y, and the bars will be called an (x, y) configuration. It is clear that a configuration exists. If a point is on some line of a configuration, it will be convenient to say that the point is of, or on, the configuration. Immediately it is seen that at least two bars of a configuration intersect (if not, the number of points on the configuration is $n\cdot n+n$, which exceeds the total number of points in the geometry).

No bar intersects two other bars. If P_i is a point of x, let a_i be the bar on P_i. Suppose that for the distinct integers l, j, and k, a_k intersects a_l and a_j. Then on any point of x not P_j, P_k, P_l, there are two lines parallel to a_k. At least one of these is not a bar. Let one, not a bar, on point P_i be c_i. Since it is not a bar, and certainly is not x, it must intersect y (if not, there would be three parallels to y on P_i, a contradiction), say at R_i. There are $n-3$ lines c_i $(i\neq j,\ k,\ l)$. The lines c_j and c_j' on P_j which are parallel to a_k are distinct from x and a_j; hence, they intersect y in points R_j and R_j'. Similarly parallels c_l, c_l' intersect y in points R_l and R_l'. We have described $n+1$ lines which are parallel to a_k, which are distinct from y, and which intersect y. Thus two of them intersect in a point on y; then there are three parallels to a_k from this point, a contradiction. This contradiction proves that at most one bar intersects any other given bar, or that bars intersect at most in pairs.

From the preceding, then, there exists a configuration, and any configuration has at least one pair of intersecting bars.

THEOREM II. *If there exists a configuration with k pairs of intersecting bars, then there exists a configuration with at most $k-1$ pairs of intersecting bars.*

Case 1. $2k=n$.[1] n must be even (and greater than 2) so $n\geqq 4$. Hence there are at least two pairs of intersecting bars in the configuration,

[1] I am indebted to the referee for the suggestion that a single theorem with two cases would suffice in place of two separate theorems in the original paper.

say a, b, c, d, where a and b intersect, as do c and d, but of course a and b are parallel to each of c and d. Suppose that a intersects b at S, and c intersects d at D. Then at each point not S of line a, there are two parallels to b. At least one of each of these intersects d (if not, the two parallels and a are three lines on a point, all parallel to d, a contradiction), but none is on d at D (if q, say, is on d at D, then q, c, and d are distinct and parallel to b, a contradiction). So on each of the $n-1$ points not S of a there is a parallel to b which is on one of the $n-1$ points not D of d; at no such point of a are both parallels to b on d (if so, then at some point of d there are two distinct parallels to b, besides d itself, a contradiction). Then at every point not S of a, there is a parallel to b which is also parallel to d. Then b is a bar of the (a, d) configuration which intersects no other bar of that configuration. If this configuration has s pairs of intersecting bars, it must be that $2s<n$, since b is not a member of any pair of intersecting bars. Hence $s\leqq k-1$, as was to be shown.

Case 2. $2k<n$. If the (r, l) configuration exists, with k pairs of intersecting bars, and $2k<n$, then the configuration has n^2+n points on it, all distinct except for the k intersections of bars; it must be that there are n^2+n-k distinct points on the configuration. There are n^2+n-1 points in the geometry, so there are $k-1$ points not on the configuration. It will be shown that there exists a configuration whose bars intersect at no other points than those $k-1$ points, and which accordingly has at most $k-1$ pairs of intersecting bars.

Since $2k<n$, not every bar of the (r, l) configuration is a member of a pair of intersecting bars. Suppose, then, that h is a bar which intersects no other. Then the (l, h) configuration certainly exists. Suppose, further, that a pair of bars s, t of this configuration intersect at a point of the (r, l) configuration. They do not intersect on l or h; the intersection must occur at a point on some bar (not h) of the (r, l) configuration. Let the point be B on bar e. The lines s and t are distinct from e, since they intersect l and e does not. It is seen that on B are three parallels to h: s, t, and e (h was given as a bar which intersected no other bar of the (r, l) configuration), the desired contradiction. The bars of the (l, h) configuration have points of intersection only at points not on the (r, l) configuration, so it is the required configuration with at most $k-1$ pairs of intersecting bars.

By induction it follows from the preceding theorem that there exists a configuration with no intersecting bars (a contradiction). Thus the axioms are inconsistent for $m=2$, $n>2$.

REFERENCES

1. Abraham J. Barshop, *A miniature mathematical science,* Brooklyn College Math. Mirror, Annual Bulletin of the Mathematics Club of Brooklyn College **7** (1939), 14–17.

2. Lawrence M. Graves, *A finite Bolyai-Lobachevsky plane,* Amer. Math. Monthly **69** (1962), 130–132.

3. L. Szamkolowicz, *On the problem of existence of finite regular planes,* Colloq. Math. **9** (1962), 245–250.

4. Bernard J. Topel, *Bolyai-Lobachevski planes with finite lines,* pp. 40–42, Notre Dame Univ. Press, Notre Dame, Ind., 1944 (Reports of a Mathematical Colloquium, Ser. 2, no. 5/6).

OREGON STATE UNIVERSITY

ON THE SECOND COHOMOLOGY GROUP OF A KAEHLER MANIFOLD OF POSITIVE CURVATURE

R. L. BISHOP[1] AND S. I. GOLDBERG[2]

1. Introduction. A. Andreotti and T. Frankel proved that a 4-dimensional compact Kaehler manifold of strictly positive sectional curvature is analytically homeomorphic with complex projective 2-space, and the latter conjectured that this is true in all dimensions [2], [5]. In this note, evidence is given in support of this conjecture. In fact, the following theorem is proved.

THEOREM 1. *Let M be a complete Kaehler manifold of strictly positive curvature. Then, the second Betti number $b_2(M, R)$ is* 1, *i.e.,* $\dim H^2(M, R) = 1$ *where $H^i(M, R)$ is the ith cohomology group of M with real coefficients.*

Since the 2-form defined by the Ricci tensor of M is closed, and in fact, is also co-closed if the scalar curvature is a constant, we obtain immediately

COROLLARY (KOSZUL-MATSUSHIMA [7], [9]). *A homogeneous Kaehler manifold of strictly positive curvature is an Einstein space, i.e., a space of constant mean curvature.*

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