PARALLEL VECTOR FIELDS

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We speak of a parallel vector field if parallel transport is independent of the path of propagation [1, p. 239]. Riemannian spaces which are flat offer a simple example for the existence of such fields [1, p. 239]. We wish to generalize the concept of parallel displacement of a vector in a Riemannian space. In our case, spaces of constant negative curvature are the most conspicuous to admit parallel vector fields.

DEFINITION. *Let the direction of a vector field at any point be that of the unit vector* $\mathbf{V}$. *The field is said to be parallel if*

$$V_{i,j} = L(g_{ij} - V_iV_j), \qquad L \neq 0. \tag{1}$$

THEOREM. *For* $n>2$, *the system of equations* (1) *is integrable and arbitrary initial values of* V_i *may be prescribed to determine the field uniquely if and only if the Riemannian curvature* K *is constant and*

$$-L_{,j} = (L^2 + K)V_j. \tag{2}$$

Covariant differentiation of (1) yields the usual integrability conditions. With the aid of a Ricci identity [1, p. 215] we write

$$L_{,k}(g_{ij} - V_iV_j) - L_{,j}(g_{ik} - V_iV_k) - L^2(g_{hj}g_{ik} - g_{hk}g_{ij})V^h = R_{hijk}V^h. \tag{3}$$

If the Riemannian curvature is constant [1, p. 236],

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}). \tag{4}$$

Thus if K is constant and (2) holds, conditions (3) are identically satisfied. This shows sufficiency for our theorem. Also, if $n=2$, relation (4) is still valid without K necessarily being constant and (2) again insures sufficiency.

We now prove the necessity part of our theorem. Multiplication of (3) by $W^iV^jW^k$ leads to

$$-(V^rL_{,r} + L^2)(g_{hj}g_{ik} - g_{hk}g_{ij})V^hW^iV^jW^k = R_{hijk}V^hW^iV^jW^k.$$

This shows that the Riemannian curvature in the direction of the unit vectors $\mathbf{V}$, $\mathbf{W}$ [1, p. 236] is $-(V^jL_{,j}+L^2)$. If, at a given point, solutions of (1) exist in all directions, $\mathbf{W}$ also being one of them, then the curvature would be given equally well by $-(W^jM_{,j}+M^2)$, where M indicates the scalar factor belonging to $\mathbf{W}$. Proceeding from here

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we use Schur's theorem to arrive at the conclusion that the curvature must be constant. But then, using (4) again, we can multiply (3) by g^{ik}, which, dividing by $n-2$, gives (2).

As a side light, differentiating (2) covariantly, we find the Laplacian

$$g^{ij}L_{,ij} = -L(L^2 + K)(n - 3).$$

This insinuates a query about a scalar L whose Laplacian vanishes. In case $n > 3$ we have $L^2 + K = 0$.

In general, if we specify $L_{,j} = 0$, then $L^2 + K = 0$ is a necessary and sufficient condition for the existence of solutions. This is seen to be true also for $n = 2$. To find a result of geometric interest, assuming L constant, let θ be the angle between V and a geodesic, that is $V_i d_s x^i = \cos\theta$. When V undergoes a parallel displacement along the geodesic, we easily obtain using (1) $d_s\theta = -L \sin\theta$. Hence, with the proper choice of the integration constant, we get the well-known formula for the angle of parallelism in hyperbolic space, namely $\log \tan (\theta/2) = -Ls$.

REFERENCE

1. T. J. Willmore, *An introduction to differential geometry*, Oxford Univ. Press, Oxford, 1959.

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