We speak of a parallel vector field if parallel transport is independent of the path of propagation [1, p. 239]. Riemannian spaces which are flat offer a simple example for the existence of such fields [1, p. 239]. We wish to generalize the concept of parallel displacement of a vector in a Riemannian space. In our case, spaces of constant negative curvature are the most conspicuous to admit parallel vector fields.

**Definition.** Let the direction of a vector field at any point be that of the unit vector \( V \). The field is said to be parallel if

\[
V_{i,j} = L(g_{ij} - V_i V_j), \quad L \neq 0.
\]

**Theorem.** For \( n > 2 \), the system of equations (1) is integrable and arbitrary initial values of \( V_i \) may be prescribed to determine the field uniquely if and only if the Riemannian curvature \( K \) is constant and

\[
-L_{i,j} = (L^2 + K) V_j.
\]

Covariant differentiation of (1) yields the usual integrability conditions. With the aid of a Ricci identity [1, p. 215] we write

\[
L_{i,k}(g_{ij} - V_i V_j) - L_{j,i}(g_{ik} - V_i V_k) - L^2(g_{ik} g_{ia} - g_{ik} g_{ia}) V^k = R_{kij} V^i.
\]

If the Riemannian curvature is constant [1, p. 236],

\[
R_{kij} = K(g_{ik} g_{ja} - g_{ik} g_{ja}).
\]

Thus if \( K \) is constant and (2) holds, conditions (3) are identically satisfied. This shows sufficiency for our theorem. Also, if \( n = 2 \), relation (4) is still valid without \( K \) necessarily being constant and (2) again insures sufficiency.

We now prove the necessity part of our theorem. Multiplication of (3) by \( W^i V^j W^* \) leads to

\[
-(V^r L_r + L^2)(g_{ij} g_{ak} - g_{ik} g_{aj}) V^k W^i V^j W^* = R_{kij} V^k W^i V^j W^*.
\]

This shows that the Riemannian curvature in the direction of the unit vectors \( V, W \) [1, p. 236] is \( -(V^r L_r + L^2) \). If, at a given point, solutions of (1) exist in all directions, \( W \) also being one of them, then the curvature would be given equally well by \( -(W^r M_r + M^2) \), where \( M \) indicates the scalar factor belonging to \( W \). Proceeding from here

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we use Schur's theorem to arrive at the conclusion that the curvature must be constant. But then, using (4) again, we can multiply (3) by $g^{ik}$, which, dividing by $n-2$, gives (2).

As a side light, differentiating (2) covariantly, we find the Laplacian

$$g^{ij}L_{ij} = - L(L^2 + K)(n - 3).$$

This insinuates a query about a scalar $L$ whose Laplacian vanishes. In case $n > 3$ we have $L^2 + K = 0$.

In general, if we specify $L_{,i} = 0$, then $L^2 + K = 0$ is a necessary and sufficient condition for the existence of solutions. This is seen to be true also for $n = 2$. To find a result of geometric interest, assuming $L$ constant, let $\theta$ be the angle between $V$ and a geodesic, that is $V_{,i}x^i = \cos \theta$. When $V$ undergoes a parallel displacement along the geodesic, we easily obtain using (1) $d\theta = -L \sin \theta$. Hence, with the proper choice of the integration constant, we get the well-known formula for the angle of parallelism in hyperbolic space, namely $\log \tan (\theta/2) = -Ls$.

**Reference**


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