A NOTE ON "THE COMPARABILITY GRAPH OF A TREE"

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1. **Introduction.** The purpose of this note is to provide a much simpler and more illuminating proof of the main result of the author's previous paper [7], and also to deduce some interesting corollaries of this result. Our general terminology and notation are the same as [7], to which we refer the reader for any definitions not given here.

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If (G, R) is an unoriented graph, we shall refer to the elements of G as *vertices*. If x and y are vertices with $x\ R\ y$, then the unordered pair $\{x, y\}$ will be called an *edge* of (G, R). The notation (x, y) will denote an ordered pair.

We say that a relation T on G is an *orientation* of (G, R) if and only if (i) (G, T) is an oriented graph, and (ii) $x\ R\ y$ if and only if $x\ T\ y$ or $y\ T\ x$, for all $x, y \in G$. An unoriented graph which possesses a transitive orientation will be called a *comparability graph.* Thus, in slightly different language, an unoriented graph (G, R) is a comparability graph if and only if there exists a partial ordering $<$ of the vertices of G, such that $x\ R\ y$ if and only if $x<y$ or $y<x$. In this case, (G, R) is called the *comparability graph of the partially ordered set* $(G, <)$.

Recently A. Ghouila-Houri [2] and P. C. Gilmore and A. J. Hoffman [3] have given necessary and sufficient conditions for an unoriented graph to be a comparability graph, but we shall not make use of these conditions here.

2. **A preliminary result.** We shall first show that the proof that a graph possesses a transitive orientation may always be reduced to the finite case. While such a result is proved in [3], we shall give another proof based on the following well-known theorem of R. Rado [4], [6].

THEOREM 1 (RADO). *Let I be an arbitrary index set, and let $\{X_i: i \in I\}$ be a family of finite sets. For each finite subset A of I, let f_A be a choice function for the family $\{X_i: i \in A\}$. Then there exists a choice function f for $\{X_i: i \in I\}$ such that, whenever A is a finite subset of I, there is a finite set B, $A \subseteq B \subseteq I$, with $f(i) = f_B(i)$ for all $i \in A$.*

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We now prove

THEOREM 2. *If every finite subgraph of an unoriented graph* (G, R) *admits a transitive orientation, then so does* (G, R).

PROOF. Let E be the set of all edges of (G, R). If $e=\{x, y\}$ is a member of E, let Z_e be the set whose elements are the ordered pairs (x, y) and (y, x): i.e., $Z_e=\{(x, y), (y, x)\}$. For each finite subset A of E, let f_A be a choice function for $\{Z_e: e\in A\}$ such that the orientation defined by the set of ordered pairs $\{f_A(e): e\in A\}$ is a transitive orientation of the subgraph determined by A. Then, by Theorem 1, there exists a choice function f for $\{Z_e: e\in E\}$ such that, whenever A is a finite subset of E, there is a finite set B, $A\subseteq B\subseteq E$, with $f(e)=f_B(e)$ for all $e\in A$. Define $T=\{f(e): e\in E\}$. Clearly T is an orientation of (G, R). To show that T is transitive, suppose that $x\ T\ y$ and $y\ T\ z$. Let A be the set of edges $\{\{x, y\}, \{y, z\}, \{x, z\}\}$. There is a finite set of edges B, with $A\subseteq B$, such that $f(e)=f_B(e)$ for all $e\in A$. The orientation determined by f_B is transitive on the subgraph determined by B. Hence we must have $f(\{x, z\})=f_B(\{x, z\})=(x, z)$, and thus $x\ T\ z$.

3. **The comparability graph of a tree.** In [7] we said that an unoriented graph (G, R) has the *diagonal property* if and only if whenever x_1, x_2, x_3, and x_4 are distinct vertices of G with $x_1\ R\ x_2\ R\ x_3\ R\ x_4$, then $x_1\ R\ x_3$ or $x_2\ R\ x_4$. A graph with the diagonal property will be called a *D-graph.* The following theorem was the main result of [7].

THEOREM 3. *An unoriented graph* (G, R) *is the comparability graph of a tree if and only if* (G, R) *is a D-graph.*

The existence of a transitive orientation on a D-graph G follows rather simply from the criteria of Ghouila-Houri or Gilmore and Hoffman. However, these theorems do not enable us to conclude that a transitive orientation may be selected for G so as to make the resulting partially ordered set a *tree.* This result was proved in [7] by a rather complicated transfinite induction. We now give a much simpler proof, making use of Theorem 2 and the following lemma. Let us say that a vertex c of a graph (G, R) is a *central point* if and only if $c\ R\ x$ for all $x\in G$, $x\neq c$.

LEMMA. *A finite connected D-graph* (G, R) *has at least one central point.*

PROOF. Let c be a vertex of argest degree of (G, R) and let a_0, $a_1, \cdots, a_m$ be all the vertices joined to c by edges. Suppose that G

has vertices other than $c, a_0, a_1, \cdots, a_m$. Then by the connectedness of (G, R) there exists such a vertex b which is joined to one of $a_0, a_1, \cdots, a_m$, say a_0. For any $i=1, 2, \cdots, m$, we have $a_i\ R\ c\ R\ a_0\ R\ b$, but $c\ \tilde{R}\ b$: hence by the diagonal property we must have $a_i\ R\ a_0$. But then a_0 is a vertex of greater degree than c: a contradiction. Therefore $c, a_0, a_1, \cdots, a_m$ comprise all the vertices of G and c is a central point.

The reader may easily construct examples to show that the above lemma need not hold if G is infinite.

We now prove Theorem 3. We consider only the sufficiency of the diagonal property (a proof of the necessity of this condition is given in [7]). By Theorem 2 we may assume that G is finite. We may also assume that (G, R) is connected; for, if it is not, then it suffices to prove the theorem for each connected component. Our proof is by induction. The number of vertices of G will be denoted by $|G|$. Let us assume that Theorem 3 is true whenever $|G| < n$; and let (G, R) be a connected D-graph with $|G| = n$. Let c be a central point of G. By our inductive hypothesis, $G - \{c\}$ possesses a transitive orientation S, such that $(G - \{c\}, S)$ is a tree. (If $G - \{c\}$ is not connected, we may apply the inductive hypothesis to each connected component.) We adjoin c to $G - \{c\}$ as a "least" element. More precisely, we define a relation T on G by

$$T = S \cup \{(c, x) : x \in G - \{c\}\}.$$

It is clear that T is a transitive orientation of (G, R). Furthermore, since $G - \{c\}$ is a tree with respect to the orientation S, it follows that (G, T) is also a tree.

4. The complement of a *D*-graph. The graph $(G, \tilde{R})$ will be called the *complement* of (G, R). The reader may easily verify that a graph (G, S) is the complement of a D-graph if and only if whenever x_1, x_2, x_3, and x_4 are distinct vertices of G with $x_1\ S\ x_2\ S\ x_3\ S\ x_4$, then one of the relations $x_1\ S\ x_3$, $x_2\ S\ x_4$, or $x_1\ S\ x_4$ must hold. We shall prove the following theorem.

THEOREM 4. *If (G, R) is a D-graph, then $(G, \tilde{R})$ is a comparability graph.*

PROOF. We may assume that G is finite, and let us assume that the theorem holds whenever $|G| < n$. Let (G, R) be a D-graph with $|G| = n$ and suppose that (G, R) is connected. Then (G, R) has a central point c, which is an isolated point of $(G, \tilde{R})$. By our inductive hypothesis, $(G - \{c\}, \tilde{R})$ has a transitive orientation T. Since there exist no edges from c to any vertex of $(G, \tilde{R})$, it follows that T also defines a transitive orientation of $(G, \tilde{R})$. Thus the theorem is true whenever (G, R) is connected.

If (G, R) is not connected, let $(G_1, R_1), \cdots, (G_m, R_m)$ be its connected components. By the preceding argument each graph $(G_i, \tilde{R}_i)$, for $i=1, \cdots, m$, possesses a transitive orientation. But in the graph $(G, \tilde{R})$ every vertex of G_i is joined by an edge to every vertex of G_j, for $i \neq j$. Hence it is clear that the existence of a transitive orientation on each $(G_i, \tilde{R}_i)$ implies that $(G, \tilde{R})$ may be transitively oriented.

The concept of the dimension of a partially ordered set has been introduced by Dushnik and Miller [1]. We refer the reader to [1] and to Ore [5] for the relevant definitions. We now have the following corollary of Theorem 4.

THEOREM 5. *The dimension of a tree is* $\leqq 2$.

PROOF. If (G, R) is the comparability graph of the partially ordered set $(G, <)$, and if $(G, \tilde{R})$ is also a comparability graph, then Dushnik and Miller [1, Theorem 3.61] have shown that the dimension of the partially ordered set $(G, <)$ is $\leqq 2$. Theorem 5 thus follows immediately from Theorem 4.

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