ON SUBSPACES OF THE SPACE (m)

J. COPPING

In this note, m, c, and c_0 denote the Banach spaces of bounded, convergent, and null sequences, with the norm

$$\|x\| = \sup_n |s_n| \qquad (x = \{s_n\} \in m).$$

THEOREM. *If A is a given matrix summability method, X_0 the subspace of m consisting of all bounded sequences summed to zero by A, and c_0 a proper subspace of X_0, then there exists no bounded projection of X_0 onto c_0.*

COROLLARY. *If X is the space of all bounded sequences summed by A, and c a proper subspace of X, then there exists no bounded projection of X onto c.*

The corollary was conjectured by Wilansky [**6**, p. 250]; relevant known results are that X_0 and X are *nonseparable* ([**2**, 3.6.2] and [**1**, pp. 97–99]), that there exists a bounded projection onto c (or c_0) of any *separable* subspace of m containing c (or c_0) ([**3**, Theorem 2.2] and [**5**, Theorem 5]), and that there exists no bounded projection of m onto c or c_0 ([**4**, p. 539] and [**5**, p. 945]).

To prove the theorem, observe that by [**2**, 3.6], there exists a closed subspace Y_0 of X_0, a bounded linear operator T from m onto Y_0, and a strictly increasing sequence $\{m_p\}$ of positive integers, such that for $p=1, 2, \cdots$,

$$(1) \qquad t_{m_p} = u_p \qquad \text{for all } \{u_n\} \in m, \quad \text{where} \quad T(\{u_n\}) = \{t_n\}.$$

The construction given in [**2**, 3.6] for this operator T ensures also that $T(c_0) \subseteq c_0$.

Now suppose if possible that there exists a bounded projection P of X_0 onto c_0. Let Q_1: $Y_0 \to c_0$ be the restriction of P to Y_0, and define the bounded linear operator $T_1 = Q_1 T$: $m \to c_0$. Since $T(c_0) \subseteq c_0 \cap Y_0$,

$$(2) \qquad T_1(\{u_n\}) = T(\{u_n\}) \qquad \text{when } \{u_n\} \in c_0.$$

Define a bounded linear operator R: $c_0 \to c_0$ by

$$(3) \qquad v_p = s_{m_p} \qquad \text{for all } \{s_n\} \in c_0, \quad \text{where} \quad R(\{s_n\}) = \{v_n\},$$

for $p=1, 2, \cdots$. Let Q_2 be the restriction of R to the range of T_1,

Received by the editors August 13, 1963.

and define $T_2 = Q_2 T_1 \colon m \rightarrow c_0$. By (1), (2), and (3), if $\{u_n\} \in c_0$, and $T_2(\{u_n\}) = \{v_n\}$, then $u_p = v_p$ for $p = 1, 2, \cdots$; thus T_2 is a bounded projection of m onto c_0. This contradicts the result of Sobczyk [5, p. 945], and the theorem is proved.

For the corollary, let $(a_{n,k})$ be the matrix of the method A, and define a method B by the matrix $(b_{n,k})$, where

$$b_{n,k} = a_{n,k} - \lim_{n \rightarrow \infty} a_{n,k};$$

let

$$\lambda(B) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{n,k},$$

and let X_0' be the space of bounded sequences summed to zero by B. That c_0 is a proper subspace of X_0' follows from the hypothesis that c is a proper subspace of X when $\lambda(B) \neq 0$, and follows from [1, p. 97] when $\lambda(B) = 0$. By the theorem, and since $X_0' \subseteq X$, there exists no bounded projection of X onto c_0, but by [5, p. 938], there exist bounded projections of c onto c_0. The corollary is thus proved.

References

1. R. P. Agnew, *Convergence fields of methods of summability*, Ann. of Math. (2) 46 (1945), 93–101.

2. S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. 14 (1954), 129–160.

3. R. D. McWilliams, *On projections of separable subspaces of (m) onto (c)*, Proc. Amer. Math. Soc. 10 (1959), 872–876.

4. R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), 516–541.

5. A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. 47 (1941), 938–947.

6. A. Wilansky, *Topological divisors of zero and Tauberian theorems*, Trans. Amer. Math. Soc. 113 (1964), 240–251.

Exeter University, Exeter, England