SOME EMBEDDING THEOREMS[1]

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1. **Introduction.** It is well known that there exists a homeomorphism between two Q-spaces (=realcompact spaces) X and Y if and only if there exists an isomorphism between $C(X)$ and $C(Y)$, their rings of real-valued, continuous functions. This suggests the problem of finding algebraic conditions relating $C(X)$ and $C(Y)$ which are both necessary and sufficient for embedding Y in X. Our investigations in this direction have led to the consideration of three types of ring homomorphisms. Before we define these, let us recall that an ideal M of a ring A is a real ideal if A/M is isomorphic to R, the field of real numbers. We shall refer to the intersection of a collection of real ideals as a δ-real ideal, and we say that a subring B of a ring A is δ-dense in A if for every pair M_1 and M_2 of δ-real ideals of A with $M_1 - M_2 \neq \emptyset$, $M_1 - M_2$ contains an element of B.

DEFINITION (1.1). A homomorphism from a ring A into a ring B is a δ-homomorphism if it is nontrivial and the image of A is δ-dense in B.

We shall refer to a set of elements of a ring as subreal if it is contained in a real ideal of the ring.

DEFINITION (1.2). A δ-homomorphism is a δF-homomorphism if the image of every real ideal containing the kernel is subreal.

DEFINITION (1.3). A δ-homomorphism from a ring A into a ring B is a δG-homomorphism if for every real ideal M of A whose image is subreal, there exists an element $a \notin M$ such that the image of every real ideal not containing a is subreal.

The four main results of this paper are given in Theorems (2.3), (2.5), (2.6) and (2.7). From these it follows that a Q-space Y can be embedded in a Q-space X if and only if there exists a δ-homomorphism from $C(X)$ into $C(Y)$ and that Y can be embedded in X as a closed (open, dense) subset if and only if there exists a δF-homomorphism (δG-homomorphism, δ-isomorphism) from $C(X)$ into $C(Y)$.

2. **The embedding theorems.** It will be assumed that all topological spaces discussed here are *completely regular* and *Hausdorff*.

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[1] The results in this paper form a portion of the author's doctoral dissertation written at the Pennsylvania State University under the direction of Professor S. Mrówka, to whom the author wishes to express his appreciation.

THEOREM (2.1). *Let X be a Q-space, Y an arbitrary space, and ϕ a homomorphism from $C(X)$ into $C(Y)$. Then ϕ has the property that*

(2.1.1) *for every $g \in C(Y)$ and $y \notin Z(g)$,*[2] *there exists an $f \in C(X)$ such that $y \notin Z(\phi f)$ and $Z(g) \subset Z(\phi f)$*

if and only if there is a homeomorphism h from Y into X such that $\phi f = f \circ h$ for all $f \in C(X)$.

PROOF. (*Sufficiency*). Suppose $\phi f = f \circ h$ for all $f \in C(X)$ and that $y \notin Z(g)$ where $g \in C(Y)$. Then $h(y) \notin \mathrm{Cl}_X(h[Z(g)])$ and there exists a function $f \in C(X)$ which vanishes on $\mathrm{Cl}_X(h[Z(g)])$ but not at $h(y)$. It follows that $y \notin Z(\phi f)$ and $Z(g) \subset Z(\phi f)$.

(*Necessity*). Now let ϕ be a homomorphism from $C(X)$ into $C(Y)$ satisfying (2.1.1). We define a mapping h from Y into X as follows. Let $y \in Y$ be given. Then $\phi_y: C(X) \to R$ is a homomorphism where ϕ_y is given by $\phi_y f = (\phi f)(y)$. Moreover, since $y \notin Z(1)$,[3] (2.1.1) guarantees the existence of an element $f \in C(X)$ such that $y \notin Z(\phi f)$. Hence $\phi_y f \neq 0$, i.e., ϕ_y is a nontrivial homomorphism into R. Since X is a Q-space, there exists a unique $x \in X$ such that for each $f \in C(X)$, $\phi_y f = f(x)$. We define $h(y) = x$. Then for any $f \in C(X)$, $\phi f = f \circ h$ and it remains for us to show that h is a homeomorphism.

For any Z-set $Z(f)$ of X, we have $h^{\leftarrow}[Z(f)] = (f \circ h)^{\leftarrow}(0) = Z(\phi f)$ and since the Z-sets of X form a basis for the closed sets, we conclude that h is continuous.

Now if y_1 and y_2 are two distinct elements of Y, there is a function $g \in C(Y)$ such that $g(y_1) = 0$ and $g(y_2) = 1$. Then by (2.1.1) there is an $f \in C(X)$ such that $y_2 \notin Z(\phi f)$ and $y_1 \in Z(\phi f)$. Hence $f(h(y_2)) = \phi f(y_2) \neq 0$ while $f(h(y_1)) = \phi f(y_1) = 0$. Therefore $h(y_1) \neq h(y_2)$ and h is one-to-one.

To conclude the proof, we need only show $h^{\leftarrow}$ is a continuous mapping from $h[Y]$ onto Y. Choose $g \in C(Y)$ and $y \notin Z(g)$. Let f denote the function whose existence is guaranteed by (2.1.1). It follows that $h(y) \in h[Y] \cap [X - Z(f)] \subset h[Y - Z(g)]$. Hence h is an open mapping which implies $h^{\leftarrow}$ is continuous.

If X is any topological space and F is a closed subset of X, then $\{f \in C(X): F \subset Z(f)\}$, which we will denote hereafter by M_F, is an ideal of $C(X)$. Moreover, $M_F = \bigcap \{M_x: x \in F\}$ and hence is a δ-real ideal. In the case of Q-spaces, the converse is also true. That is, if M is a δ-real ideal of $C(X)$ where X is a Q-space, there is a unique closed subset $F \subset X$ such that $M = M_F$. In fact, X is a Q-space if

[2] $Z(g)$ denotes the set of points on which g vanishes and is referred to as a Z-set.

[3] For any real number k, k denotes the function which maps every point of the space into k.

and only if every δ-real ideal of $C(X)$ is of the form M_F for some closed subset F of X. We shall use these facts without explicit mention.

THEOREM (2.2). *Let X be an arbitrary topological space and Y a Q-space. The following statements concerning a homomorphism ϕ from $C(X)$ into $C(Y)$ are equivalent.*

(2.2.1) *For every $g \in C(Y)$ and $y \notin Z(g)$, there exists an $f \in C(X)$ such that $y \notin Z(\phi f)$ and $Z(g) \subset Z(\phi f)$.*

(2.2.2) *ϕ is a δ-homomorphism.*

(2.2.3) *The image of $C(X)$ separates points and closed sets and is contained in no δ-real ideal of $C(Y)$.*

(2.2.4) *ϕ is the identity mapping on constant functions and the image of $C(X)$ separates points and closed sets.*[4]

PROOF. (2.2.1)$\Rightarrow$(2.2.2). Let M_F and M_H be two δ-real ideals of $C(Y)$ with $M_F - M_H \neq \emptyset$. Then $H \not\subset F$ and there is an element $y \in H - F$ and a function $g \in C(Y)$ which vanishes on F but not at y. According to (2.2.1), there exists an $f \in C(X)$ such that $y \notin Z(\phi f)$ and $Z(g) \subset Z(\phi f)$. Hence $\phi f \in M_F - M_H$.

(2.2.2)$\Rightarrow$(2.2.3). Let M_F be any δ-real ideal of $C(Y)$. If $F = Y$, then $M_F = (0)$ and since ϕ is nontrivial, the image of $C(X)$ cannot be contained in M_F. On the other hand, if $F \neq Y$, choose $y \notin F$. Then $M_y - M_F \neq \emptyset$ and hence must contain an element of the image of $C(X)$. In either case, $\phi[C(X)]$ is not contained in M_F. Now then, if $y \notin F$, $M_F - M_y \neq \emptyset$ and $\phi f \in M_F - M_y$ for some $f \in C(X)$. Hence $\phi f(y) \notin \mathrm{Cl}(\phi f[F])$, i.e., $\phi[C(X)]$ separates points and closed sets.

(2.2.3)$\Rightarrow$(2.2.4). Consider $\phi 1$ and suppose there exists a point $y \in Y$ such that $\phi 1(y) = 0$. Then for any $f \in C(X)$, $\phi f(y) = [\phi f(y)][\phi 1(y)] = 0$ which implies $\phi[C(X)] \subset M_y$, a contradiction since the image of $C(X)$ is contained in no δ-real ideal. Hence $\phi 1(y) = 1$ for every $y \in Y$, i.e., $\phi 1 = 1$. By induction, it follows that $\phi n = n$ for every positive integer n and from this it follows that $\phi r = r$ for every rational number r. Using the fact that the rationals are dense in R and that ϕ is also a lattice homomorphism, it can be shown that $\phi k = k$ for every real number k.

(2.2.4)$\Rightarrow$(2.2.1). Suppose $g \in C(Y)$ and $y \notin Z(g)$. Then since $\phi[C(X)]$ separates points and closed sets, there is a function $f \in C(X)$ such that $k = \phi f(y) \notin \mathrm{Cl}(\phi f[Z(g)])$. Let $f_1 = f - k$. Since ϕ is the identity mapping on constant functions, $\phi f_1(y) = 0$ and $\phi f_1(y) \notin \mathrm{Cl}(\phi f_1[Z(g)])$. Hence there exists a positive number ϵ such that

[4] S. Mrówka pointed out that (2.2.4) is equivalent to (2.2.1).

$|\phi f_1(p)| > \epsilon$ for $p \in Z(g)$. Now let $f_2 = 0 \vee [\epsilon - |f_1|]$. Since $\phi|f_1| = |\phi f_1|$, $\phi f_2(y) \neq 0$ and $Z(g) \subset Z(\phi f_2)$.

From Theorems (2.1) and (2.2), we get

THEOREM (2.3). *Let X and Y be Q-spaces. A homomorphism from $C(X)$ into $C(Y)$ is a δ-homomorphism if and only if there exists a homeomorphism h from Y into X such that $\phi f = f \circ h$ for all $f \in C(X)$.*

The following lemma is needed.

LEMMA (2.4). *Let X and Y be topological spaces and h a continuous function from Y into X. Define a homomorphism ϕ from $C(X)$ into $C(Y)$ by $\phi f = f \circ h$. Then for real ideals M_x and M_y of $C(X)$ and $C(Y)$, respectively, $\phi[M_x] \subset M_y$ if and only if $h(y) = x$.*

PROOF Suppose $h(y) = x$. Then for $f \in M_x$, $\phi f(y) = f(h(y)) = f(x) = 0$. Thus $\phi[M_x] \subset M_y$. Conversely, suppose $\phi[M_x] \subset M_y$ and let $f \in C(X)$. Then $f - f(x) \in M_x$ and $\phi f - f(x) \in M_y$. Hence $f(h(y)) = \phi f(y) = f(x)$ for every $f \in C(X)$ which implies that $h(y) = x$.

THEOREM (2.5). *Let X and Y be Q-spaces. A homomorphism from $C(X)$ into $C(Y)$ is a δF-homomorphism if and only if there exists a homeomorphism h from Y into X such that $\phi f = f \circ h$ for all $f \in C(X)$, and $h[Y]$ is a closed subset of X.*

PROOF (*Sufficiency*). Let M be a real ideal of $C(X)$ which contains the kernel, $K(\phi)$, of ϕ. Then $M = M_x$ for some $x \in X$. Moreover, since $h[Y]$ is closed, $x \in h[Y]$ since otherwise there would exist a function $f \in C(X)$ vanishing on $h[Y]$ but not at x. This would imply $f \in K(\phi) - M_x$, a contradiction. Then $x = h(y)$ for some $y \in Y$ and by the previous lemma, $\phi[M_x] \subset M_y$, that is, the image of M is subreal in $C(Y)$.

(*Necessity*) Now suppose ϕ is a δF-homomorphism. By Theorem (2.3), there is a homeomorphism h from Y into X such that $\phi f = f \circ h$ for all $f \in C(X)$. It remains to show that $h[Y]$ is a closed subset of X. Choose $x \in \mathrm{Cl}_X(h[Y]) = F$. Then $K(\phi) = M_F$ and $M_F \subset M_x$. Therefore there is a real ideal M_y of $C(Y)$ such that $\phi[M_x] \subset M_y$, which, by Lemma (2.4), implies $x = h(y)$. Thus $x \in h[Y]$ and we conclude that $h[Y] = \mathrm{Cl}_X(h[Y])$.

THEOREM (2.6). *Let X and Y be Q-spaces. A homomorphism ϕ from $C(X)$ into $C(Y)$ is a δG-homomorphism if and only if there exists a homeomorphism h from Y into X such that $\phi f = f \circ h$ for all $f \in C(X)$, and $h[Y]$ is an open subset of X.*

PROOF (*Sufficiency*). Let M_x be a real ideal of $C(X)$ whose image is subreal in $C(Y)$. Hence for some $y \in Y$, $\phi[M_x] \subset M_y$. Again using

Lemma (2.4), we have $h(y)=x$. Then $x\notin X-h[Y]$ which is a closed subset of X and there is an $f\in C(X)$ which vanishes on $X-h[Y]$ but not at x. Then $f\notin M_x$ and it follows that the image of every real ideal not containing f is subreal.

(*Necessity*) Now suppose ϕ is a δG-homomorphism and $x\in h[Y]$. Then $x=h(y)$ for some $y\in Y$ and $\phi[M_x]\subset M_y$. Therefore there exists a function $f\in C(X)$ such that $f\notin M_x$ and the image of every real ideal not containing f is subreal. From this it follows that $x\in[X-Z(f)]\subset h[Y]$ which in turn implies that $h[Y]$ is open.

Finally, we note that if h is a continuous function from Y into X, the homomorphism ϕ given by $\phi f=f\circ h$ is an isomorphism if and only if $h[Y]$ is dense in X. This, in conjunction with Theorem (2.3), gives:

THEOREM (2.7). *Let X and Y be Q-spaces. A homomorphism ϕ from $C(X)$ into $C(Y)$ is a δ-isomorphism if and only if there exists a homeomorphism h from Y into X such that $\phi f=f\circ h$ for all $f\in C(X)$ and $h[Y]$ is a dense subset of X.*

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