By a K-R manifold we mean an $n$-manifold with boundary $M^n$ such that $\text{Int } M^n = E^n$ and $\text{Bd } M^n = E^{n-1}$; $\text{Int } M^n$ and $\text{Bd } M^n$ are the interior and boundary of $M^n$ respectively. Both Cantrell [2] and Doyle [3] have shown that for $n \neq 3$, each K-R manifold is the product $E^{n-1} \times [0, 1)$. But for $n = 3$ there are infinitely many K-R manifolds which are topologically distinct as pointed out in [4] and [5]. We will investigate certain properties of these manifolds with boundary.

**Lemma 0.** Let $M^n$ be a K-R manifold. Then $M^n$ is the product $E^{n-1} \times [0, 1)$ if each compact set in $M^n$ lies in a closed $n$-cell in $M^n$.

**Proof.** The proof is simple in that $M^n$ can be represented as a union of closed $n$-cells $\bigcup C_i$ where $C_i \cap \text{Bd } M^n$ is an $(n-1)$-cell $D_i$ nicely imbedded in $\text{Bd } C_i$ and $\text{Bd } M^n$, $D_i \subset \text{Int } D_{i+1}$ and $C_i - D_i \subset \text{Int } C_{i+1}$, while $[C_{i+1} - C_i]$ is an $n$-cell. One can then construct a homeomorphism of $M^n$ onto a copy of $E^{n-1} \times [0, 1)$.

**Lemma 1.** Let $M^n$ be an $n$-manifold with boundary. If $C$ is a compact set in $M^n$ such that $C \cap \text{Bd } M^n$ lies in an open $(n-1)$-cell in $\text{Bd } M^n$, then there is a pseudo-isotopy $h_t$ of $M^n$ onto $M^n$ such that $h_t(C) \subset F \cup C'$, where $F$ is a fiber in a collar about $\text{Bd } M^n$, and $C'$ is a compact set in $\text{Int } M^n$.

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Proof. That $\text{Bd } M^n$ is collared in $M^n$ follows from [1]. Since $C \cap \text{Bd } M^n$ lies in an open $(n-1)$-cell in $\text{Bd } M^n$, there is a closed $n$-cell $P^n$ in $M^n$ such that $P^n \cap \text{Bd } M^n$ is an $(n-1)$-cell $Q^n_{n-1}$.

$C \cap \text{Bd } M^n \subset \text{Int } Q^n_{n-1}$, $[\text{Bd } P^n - Q^n_{n-1}]$ is an $(n-1)$-cell $Q^n_{n-1}$ and the set $M^n = (M^n - P^n) \cup Q^n_{n-1}$ is homeomorphic to $M^n$.

Since $C \cap \text{Bd } M^n \subset \text{Int } Q^n_{n-1}$, $C \cap Q^n_{n-1} \subset \text{Int } Q^n_{n-1}$. Thus $C \cap P^n$ lies in an $n$-cell $P^n_1$ in $P^n$, $\text{Bd } P^n_1 \cap \text{Bd } P^n$ is a pair of $(n-1)$-cells in $\text{Int } Q^n_{n-1}$ and $\text{Int } Q^n_{n-1}$. One can evidently find a pseudo-isotopy $h_t$ of $M^n$ onto $M^n$ which carries $P^n_1$ to a fiber $F$ in the collar about $\text{Bd } M^n$, while $h_t$ is fixed outside any neighborhood $U$ of $P^n_1$ and for all $t$, $h_t(P^n_1) = P^n$.

If $h_t$ is the terminal map, let $C' = h_1[(C - P^n)]$. Then $h_1(C) \subset F \cup C'$, $\rho = F \cap \text{Bd } M^n$, a point.

Theorem 1. Let $M^n$ be a 3-dimensional $K-R$ manifold, $M^n \neq E^3 \times [0, 1)$. Then there is a polygonal graph $G (G \cap \text{Bd } M^n = \rho, a \text{ point})$ in $M^n$ which lies in no closed 3-cell $J^3$ in $M^n$ such that $G - \rho \subset \text{Int } J^3$.

Proof. Let $M^n$ be given a fixed triangulation [7]. By Lemma 0 there is a compact set $C \subset M^n$ and $C$ lies in no closed 3-cell in $M^n$. We assume without loss of generality that $C \cap \text{Bd } M^n$ is a disk $D$. Since $C$ lies in no closed 3-cell in $M^n$, $C$ lies in no closed 3-cell $K$ which meets $\text{Bd } M^n$ in a disk containing $D$ in its interior while $C - D \subset \text{Int } K$.

Now by Lemma 1, $C$ can be deformed into a set of the form $h_1(C) = F \cup C'$, where $C' \subset \text{Int } M^n$ is compact and $F$ is a polygonal fiber in the collar about $\text{Bd } M^n$, $F \cap \text{Bd } M^n = \rho$. Then again there is no closed 3-cell $K$ which meets $\text{Bd } M^n$ in a disk containing $\rho$ in its interior, while $(F \cup C') - \rho \subset \text{Int } K$. For if such a 3-cell $K$ were to exist there would be a value $0 < t < 1$ such that $h_t(C) - h_t(D) \subset \text{Int } K$ and $h_t(D)$ lies interior to the disk $K \cap \text{Bd } M^n$.

Let $N^n$ be the open 3-cell obtained by attaching an open collar to $\text{Bd } M^n$ by an extension of the triangulation on $M^n$. In order to construct $G$, let $H^n$ be a polyhedral 3-cell in $\text{Int } M^n$ such that $C' \subset \text{Int } H^n$ [6]. If $g_t$ is a pseudo-isotopy of $M^n$ onto $M^n$ which is semi-linear and fixed outside a neighborhood of $H^n$ in $\text{Int } M^n$ such that $g_1(H^n) = q$, a point, $g_t | M^n - H^n$ is a homeomorphism, then $g_t(F \cup C') = G$ is a polygonal graph. If there were a closed 3-cell $J^3$ such that $\text{Int } J^3 \supset G - \rho$, one could assume that $\text{Bd } J^3$ is locally bicollared except at $\rho$.

If $J^3$ is a 3-cell in $\text{Int } J^3$ except for the point $\rho$ of $J^3$, one can shrink $J^3$ to a point $\rho$ by a pseudo-isotopy of $M^n$ onto $M^n$ which is fixed outside of $J^3$. Evidently there is a closed 3-cell $K$ in $M^n$, $K \cap \text{Bd } M^n$ is a disk with $\rho$ in its interior, $G - \rho \subset \text{Int } K$. But by the construction of $G$ it follows that $F \cup C'$ and hence $C$ must lie in a 3-cell. But this is contrary to hypothesis.
One may quickly deduce from Theorem 1 the following characterization.

**Theorem 2.** A necessary and sufficient condition that a 3-dimensional K-R manifold $M^3$ be $E^3 \times [0, 1)$ is that each graph $G$ meeting $\text{Bd} M^3$ in a point $x$ lie interior to a closed 3-cell except for $x$.

**Corollary.** Let $M^3$ be a K-R manifold of dimension 3 and let $p$ be a point of $\text{Bd} M^3$. If $M^3 \not= E^3 \times [0, 1)$, then $\text{Int} M^3 \cup p$ is not topologically the interior of a closed 3-simplex plus a point of its boundary.

**Theorem 3.** If $M^n_1$ and $M^n_2$ are 3-dimensional K-R manifolds, then $M^n_1 \times M^n_2 = E^n \times [0, 1)$ and $M^n_1 \times E^1 = E^n \times [0, 1)$.

**Proof.** By either [2] or [3], a K-R manifold $M^n$ of dimension $n \not= 3$ is $E^{n-1} \times [0, 1)$.

**References**


7. ———, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2) 69 (1959), 37–65.

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