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THE INSTITUTE FOR ADVANCED STUDY

SPECIAL n-MANIFOLDS WITH BOUNDARY[1]

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By a K-R manifold we mean an n-manifold with boundary M^n such that Int $M^n = E^n$ and Bd $M^n = E^{n-1}$; Int M^n and Bd M^n are the interior and boundary of M^n respectively. Both Cantrell [2] and Doyle [3] have shown that for $n \neq 3$, each K-R manifold is the product $E^{n-1} \times [0, 1)$. But for $n = 3$ there are infinitely many K-R manifolds which are topologically distinct as pointed out in [4] and [5]. We will investigate certain properties of these manifolds with boundary.

LEMMA 0. *Let M^n be a K-R manifold. Then M^n is the product $E^{n-1} \times [0, 1)$ if each compact set in M^n lies in a closed n-cell in M^n.*

PROOF. The proof is simple in that M^n can be represented as a union of closed n-cells $\bigcup C_i$ where $C_i \cap \text{Bd } M^n$ is an $(n-1)$-cell D_i nicely imbedded in Bd C_i and Bd M^n, $D_i \subset \text{Int } D_{i+1}$ and $C_i - D_i \subset \text{Int } C_{i+1}$, while $[C_{i+1} - C_i]^-$ is an n-cell. One can then construct a homeomorphism of M^n onto a copy of $E^{n-1} \times [0, 1)$.

LEMMA 1. *Let M^n be an n-manifold with boundary. If C is a compact set in M^n such that $C \cap \text{Bd } M^n$ lies in an open $(n-1)$-cell in* Bd M^n, *then there is a pseudo-isotopy h_t of M^n onto M^n such that $h_1(C) \subset F \cup C'$, where F is a fiber in a collar about* Bd M^n, *and C' is a compact set in* Int M^n.

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PROOF. That Bd M^n is collared in M^n follows from [1]. Since $C \cap \text{Bd } M^n$ lies in an open $(n-1)$-cell in Bd M^n, there is a closed n-cell P^n in M^n such that $P^n \cap \text{Bd } M^n$ is an $(n-1)$-cell Q_1^{n-1}, $C \cap \text{Bd } M^n \subset \text{Int } Q_1^{n-1}$, $[\text{Bd } P^n - Q_1^{n-1}]^-$ is an $(n-1)$-cell Q_2^{n-1} and the set $M_1^n = (M^n - P^n) \cup Q_2^{n-1}$ is homeomorphic to M^n.

Since $C \cap \text{Bd } M^n \subset \text{Int } Q_1^{n-1}$, $C \cap Q_2^{n-1} \subset \text{Int } Q_2^{n-1}$. Thus $C \cap P^n$ lies in an n-cell P_1^n in P^n, Bd $P_1^n \cap \text{Bd } P^n$ is a pair of $(n-1)$-cells in Int Q_1^{n-1} and Int Q_2^{n-1}. One can evidently find a pseudo-isotopy h_t of M^n onto M^n which carries P_1^n to a fiber F in the collar about Bd M^n, while h_t is fixed outside any neighborhood U of P_1^n and for all t, $h_t(P^n) = P^n$. If h_1 is the terminal map, let $C' = h_1[(C - P^n)]^-$. Then $h_1(C) \subset F \cup C^1$, $p = F \cap \text{Bd } M^n$, a point.

THEOREM 1. *Let M^3 be a 3-dimensional K-R manifold, $M^3 \neq E^2 \times [0, 1)$. Then there is a polygonal graph G ($G \cap \text{Bd } M^3 = p$, a point) in M^3 which lies in no closed 3-cell J^3 in M^3 such that $G - p \subset \text{Int } J^3$.*

PROOF. Let M^3 be given a fixed triangulation [7]. By Lemma 0 there is a compact set $C \subset M^3$ and C lies in no closed 3-cell in M^3. We assume without loss of generality that $C \cap \text{Bd } M^3$ is a disk D. Since C lies in no closed 3-cell in M^3, C lies in no closed 3-cell K which meets Bd M^3 in a disk containing D in its interior while $C - D \subset \text{Int } K$.

Now by Lemma 1, C can be deformed into a set of the form $h_1(C) = F \cup C^1$, where $C^1 \subset \text{Int } M^3$ is compact and F is a polygonal fiber in the collar about Bd M^3, $F \cap \text{Bd } M^3 = p$. Then again there is no closed 3-cell K which meets Bd M^3 in a disk containing p in its interior, while $(F \cup C^1) - p \subset \text{Int } K$. For if such a 3-cell K were to exist there would be a value $0 < t < 1$ such that $h_t(C) - h_t(D) \subset \text{Int } K$ and $h_t(D)$ lies interior to the disk $K \cap \text{Bd } M^3$.

Let N^3 be the open 3-cell obtained by attaching an open collar to Bd M^3 by an extension of the triangulation on M^3. In order to construct G, let H^3 be a polyhedral 3-cell in Int M^3 such that $C^1 \subset \text{Int } H^3$ [6]. If g_t is a pseudo-isotopy of M^3 onto M^3 which is semi-linear and fixed outside a neighborhood of H^3 in Int M^3 such that $g_1(H^3) = q$, a point, $g_1 | M^3 - H^3$ is a homeomorphism, then $g_1(F \cup C^1) = G$ is a polygonal graph. If there were a closed 3-cell J^3 such that Int $J^3 \supset G - p$, one could assume that Bd J^3 is locally bicollared except at p.

If J_1^3 is a 3-cell in Int J^3 except for the point p of J_1^3 such that $G - p \subset \text{Int } J_1^3$ one can shrink J_1^3 to a point p by a pseudo-isotopy of M^3 onto M^3 which is fixed outside of J^3. Evidently there is a closed 3-cell K in M^3, $K \cap \text{Bd } M^3$ is a disk with p in its interior, $G - p \subset \text{Int } K$. But by the construction of G it follows that $F \cup C^1$ and hence C must lie in a 3-cell. But this is contrary to hypothesis.

One may quickly deduce from Theorem 1 the following characterization.

THEOREM 2. *A necessary and sufficient condition that a 3-dimensional K-R manifold M^3 be $E^2 \times [0, 1)$ is that each graph G meeting* Bd M^3 *in a point x lie interior to a closed 3-cell except for x.*

COROLLARY. *Let M^3 be a K-R manifold of dimension 3 and let p be a point of* Bd M^3. *If $M^3 \neq E^2 \times [0, 1)$, then* Int $M^3 \cup p$ *is not topologically the interior of a closed 3-simplex plus a point of its boundary.*

THEOREM 3. *If M_1^3 and M_2^3 are 3-dimensional K-R manifolds, then $M_1^3 \times M_2^3 = E^5 \times [0, 1)$ and $M_1^3 \times E^1 = E^3 \times [0, 1)$.*

PROOF. By either [2] or [3], a K-R manifold M^n of dimension $n \neq 3$ is $E^{n-1} \times [0, 1)$.

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