TAME ARCS ON DISKS

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It is the goal of this note to show that each disk in E^3 contains a tame arc which intersects the boundary of D. In [1] Bing shows that each disk in E^3 contains many tame arcs. The reason that the arguments given in [1] do not show that each disk contains a tame arc intersecting the boundary is that a disk in E^3 need not lie on a closed surface in E^3 [7]. This difficulty can be overcome using Bing's improvement of the "side approximation theorem" [2] and a theorem of Hempel [6].

Suppose that D is a disk in E^3.

LEMMA. *If D lies on a 2-sphere in E^3 then D contains a tame arc which intersects both* Int D *and* Bd D.

PROOF. Let S be a 2-sphere in E^3 containing D. It follows from [1] that for each positive number ϵ there exists a tame 2-sphere S' such that (i) $S\cap S'$ contains a tame Sierpiński curve X and, (ii) each component of $S-X$ is of diameter less than ϵ.

Now if ϵ is chosen less than $\min\{\text{diam } D, \text{diam } (S-D)\}$ then X must intersect both D and $S-D$, and hence Bd D. It follows that D contains a tame arc which intersects both Int D and Bd D. This establishes the lemma.

THEOREM. *D contains a tame arc which intersects both* Int D *and* Bd D.

PROOF. Let $J_1, J_2, \cdots$ be a sequence of tame simple closed curves on D such that if $D_1, D_2, \cdots$ are the disks on D bounded, respectively, by $J_1, J_2, \cdots$ then $D_i \subset \text{Int } D_{i+1}$ and $\bigcup D_i = \text{Int } D$. The existence of these tame simple closed curves follows from [1]. It follows from a theorem of Hempel [6] that for each i, D_i lies on a closed surface in E^3. This is because D_i is interior to the larger disk D_{i+1}. Now, using this fact and repeatedly applying the results of [2] and the techniques of [1], there exist tame disks $D_1', D_2', \cdots$ such that

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(a) Bd $D_i' = J_i$,
(b) $D_i \cap D_i'$ is a Sierpiński curve,
(c) $D_i' \subset \text{Int } D_{i+1}'$, and
(d) $\text{Cl}[\bigcup D_i']$ is a disk bounded by Bd D.

The procedure for obtaining these disks is, roughly, as follows: a dense, null sequence of disks is removed from the interior of D_1 and each of these disks is replaced by a tame disk. The resulting tame disk is D_1'. Then, disks are removed from the annulus on D bounded by J_1 and J_2 and are replaced by tame disks to obtain D_2'. This process is continued. Care is exercised in replacing disks with tame disks so that each of D_i' and $\text{Cl}[\bigcup D_i']$ is a disk. It follows from a theorem of Gillman [4] that the disks which are removed at the ith stage need not intersect J_i. For more details on this replacing process the reader is referred to [1].

Now let D' denote $\text{Cl}[\bigcup D_i']$. Notice that $D \cap D'$ is a Sierpiński curve which contains $\text{Cl}[\bigcup J_i]$. Now D' is locally tame at each point of Int D' and it follows from [3] that there is no loss in generality in assuming that D' is locally polyhedral at each point of Int D'. It follows from [5] that D' lies on a 2-sphere in E^3.

Now by the lemma there exists a tame arc α on D' which intersects both Int D' and Bd D'. Without loss of generality we may assume that $\alpha \cap \text{Bd } D = \{P\}$. Let β be an arc in $D \cap D'$ having P for one endpoint and such that $\beta - \{P\} \subset \text{Int } D$.

Let K be a subdisk of D' such that (i) $\alpha \cup \beta \subset K$, (ii) $K \cap \text{Bd } D' = \{P\}$ and, (iii) K is locally polyhedral except at P. Then there is a 2-sphere S in E^3 such that $K \subset S$ and S is locally polyhedral except at P. But P lies on the tame arc α and it follows from [8] that S is a tame 2-sphere. Thus the arc β is tame and satisfies the conclusion of the theorem. This establishes the theorem.

Notice that the arguments given actually show that the set of points of Bd D which are accessible by tame arcs from Int D is dense in Bd D.

COROLLARY. *If* $\epsilon > 0$ *then there is a triangulation* T *of* D *such that* (i) mesh $T < \epsilon$ *and* (ii) *if* σ *is a wild* 1-*simplex of* T *then* $\sigma \subset \text{Bd } D$.

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SPECIAL n-MANIFOLDS WITH BOUNDARY[1]

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By a K-R manifold we mean an n-manifold with boundary M^n such that Int $M^n = E^n$ and Bd $M^n = E^{n-1}$; Int M^n and Bd M^n are the interior and boundary of M^n respectively. Both Cantrell [2] and Doyle [3] have shown that for $n \neq 3$, each K-R manifold is the product $E^{n-1} \times [0, 1)$. But for $n = 3$ there are infinitely many K-R manifolds which are topologically distinct as pointed out in [4] and [5]. We will investigate certain properties of these manifolds with boundary.

Lemma 0. *Let M^n be a K-R manifold. Then M^n is the product $E^{n-1} \times [0, 1)$ if each compact set in M^n lies in a closed n-cell in M^n.*

Proof. The proof is simple in that M^n can be represented as a union of closed n-cells $\bigcup C_i$ where $C_i \cap \text{Bd } M^n$ is an $(n-1)$-cell D_i nicely imbedded in Bd C_i and Bd M^n, $D_i \subset \text{Int } D_{i+1}$ and $C_i - D_i \subset \text{Int } C_{i+1}$, while $[C_{i+1} - C_i]^-$ is an n-cell. One can then construct a homeomorphism of M^n onto a copy of $E^{n-1} \times [0, 1)$.

Lemma 1. *Let M^n be an n-manifold with boundary. If C is a compact set in M^n such that $C \cap$ Bd M^n lies in an open $(n-1)$-cell in* Bd M^n, *then there is a pseudo-isotopy h_t of M^n onto M^n such that $h_1(C) \subset F \cup C'$, where F is a fiber in a collar about* Bd M^n, *and C' is a compact set in* Int M^n.

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