A DIRECT PROOF OF TWO THEOREMS ON TWO-LINE PARTITIONS

CULBRETH SUDLER, JR.

1. **Statement of results.** B. Gordon has recently proved the two following theorems [1]. Let n be a fixed positive integer and suppose that

$$(1.1) \qquad n = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j, \qquad s \leqq r,$$

a "two-line" partition of n in which the a_i and b_j are positive integers satisfying

$$(1.2) \qquad \begin{array}{ll} a_i > a_{i+1}, \quad 1 \leqq i \leqq r-1; & b_j > b_{j+1}, \quad 1 \leqq j \leqq s-1; \\ a_j \geqq b_j, \quad 1 \leqq j \leqq s, & \end{array}$$

including the case in which the b_j are absent from (1.1). Then we have

THEOREM 1. *The number of solutions of* (1.1) *and* (1.2) *is* $p(n)$, *the number of unrestricted partitions of* n.

THEOREM 2. *The number of such solutions in odd integers is* $p([n/2])$, *where* $p(0)=1$.

In Gordon's paper, these results are obtained indirectly with the aid of generating functions and a combinatorial lemma which is proved by induction, and on p. 873 the question of finding a proof by one-to-one correspondence is raised. In the present paper we obtain such a correspondence in a reasonably simple and satisfactory manner.[1] It will be seen that our work closely parallels that of [1], and at the end of §3, we outline an alternative method based on a simpler proof of the above-mentioned lemma. Otherwise, this paper is self-contained except for a few elementary concepts of partition theory explained in [2, Chapter 19].

1a. **Notation and conventions.** Unless otherwise indicated, small Roman and Greek letters with or without subscripts will denote positive integers and capital letters will denote partitions. We denote a partition A of n by

Received by the editors July 9, 1963.

[1] S. A. Burr has announced a direct proof of Gordon's theorems (Notices Amer. Math. Soc. 10 (1963), 367). To our knowledge, however, his paper has not yet appeared.

$$A = (w_1, \cdots, w_q) = (w_p \mid 1 \leqq p \leqq q),$$

where

$$(1.3) \qquad n = \sum_{p=1}^{q} w_p; \qquad w_p \geqq w_{p+1}, \qquad 1 \leqq p \leqq q-1.$$

When necessary, we write A_n to emphasize that A is a partition of n. In numerical examples we omit commas between summands of partitions and terms of sequences and write, for example, $A=(321)$ or (3 2 1) instead of (3, 2, 1).

All summation indices are subscripts whose range, such as that of p in (1.3), will be omitted when the index assumes all values allowed by the context. We make the usual convention about vacuous conditions, so that, in particular, empty sums are equal to zero. This means that there is just one partition of 0, the empty partition.

For ease of comparison we shall use the notation of [1] wherever it is relevant, including the term δ-partition for a solution of (1.1) and (1.2). As is pointed out in [1], a δ-partition may be considered as a partition with two rows of summands, the a_i forming the top row. The term "partition," otherwise unqualified, will denote an ordinary partition normalized by (1.3).

2. **A pictorial argument and reduction of the problem.** Let $P=(x_1, \cdots, x_m)$ be a partition of n, and let $(y_1, \cdots, y_\mu)$ be the conjugate to P in the graphical sense. Here we specify that P is given by the horizontal reading of its graph, as in Figure 1. We now draw

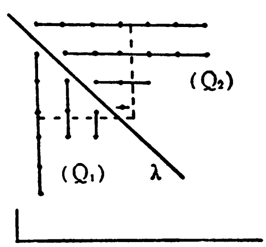

FIGURE 1

a line λ in the position shown in the figure and obtain two partitions Q_1 and Q_2 with distinct summands, represented by the sets of horizontal or vertical lines connecting the nodes. More precisely, we have

$$(2.1) \qquad Q_1 = (d_f \mid 1 \leqq f \leqq h), \qquad Q_2 = (e_g \mid 1 \leqq g \leqq \kappa),$$

where

$$(2.2)\qquad d_f = y_f - f, \qquad e_g = x_g - (g-1), \qquad \kappa - 1 \leqq h \leqq \kappa,$$

in which κ^2 is the number of nodes in the Durfee square of P. ($h=0$ and Q_1 is absent only if $P=(e_1)$.) In Figure 1 we have $P=(7754311)$, $Q_2=(7631)$, $Q_1=(632)$, $\kappa=4$, $h=3$, and the Durfee square is enclosed by the light dashed line.

Hence there is a 1-1 correspondence, given by (2.1), between the ordinary partitions P and the ordered pairs $D=[Q_1, Q_2]$, in which, for arbitrary κ, Q_1 and Q_2 are any partitions with h and κ distinct summands respectively, where $\kappa=h$ or $h+1$, and $h\geqq 0$. We call D a diagonal partition whose summands are defined to be the numbers d_f and e_g of (2.1). If D corresponds to P and $P=P_n$, then we write $D=D_n$. Here and in the sequel all correspondences are 1-1.

We now show that the number of diagonal partitions D_n with odd summands is $p([n/2])$, a fact needed for the proof of Theorem 2. Let $D_n=[Q_1, Q_2]$ be a diagonal partition of n whose summands d_f and e_g are all odd. Put

$$(2.3)\qquad \begin{aligned} d_f &= 2\delta_f - 1 \qquad (1 \leqq f \leqq h),\\ e_g &= 2\epsilon_g - 1 \qquad (1 \leqq g \leqq \kappa). \end{aligned}$$

It is clear that $h=\kappa$ or $h=\kappa-1$ according as n is even or odd. We consider these two cases separately.

(i) n even. Put

$$(2.4)\qquad \begin{aligned} R_1 &= (\delta_f - 1 \mid 1 \leqq f \leqq h), &\quad \delta_h > 1,\\ &= (\delta_f - 1 \mid 1 \leqq f \leqq h-1), &\quad \delta_h = 1, \end{aligned}$$

and $R_2=(\epsilon_g \mid 1\leqq g\leqq h)$. Since

$$\sum_1^h (\delta_f - 1) + \sum_1^h \epsilon_g = \sum_1^h (d_f - 1)/2 + \sum_1^h (e_g + 1)/2 = n/2,$$

it follows that $[R_1, R_2]$ is a diagonal partition of $n/2$.

This correspondence is clearly reversible, so the total number of partitions D_n with odd summands is equal to the number of partitions $D_{n/2}$, which is $p(n/2)$. In reversing the correspondence there is one small point to be noted. We define $[R_1, R_2]$ as before, where R_2 has h summands. If R_1 has $h-1$ summands, we then put $Q_1 =(d_1, \cdots, d_{h-1}, 1)$, i.e.,

$$(2.5)\qquad d_h = \delta_h = 1.$$

(ii) n odd. Define R_1 as in case (i) and let

$$(2.6)\qquad R_2 = (\epsilon_g \mid 2 \leqq g \leqq \kappa).$$

Note that if $\kappa=1$, both R_1 and R_2 are the empty partition. We have, for $\kappa\geqq 2$,

$$\sum_1^h (\delta_f - 1) + \sum_2^\kappa \epsilon_g = \sum_1^h (d_f - 1)/2 + \sum_2^\kappa (e_g + 1)/2$$

$$= (n - e_1)/2 + (\kappa - 1 - h)/2 = (n - e_1)/2 = (n - 1)/2 - (\epsilon_1 - 1).$$

Hence $[R_1, R_2]$ corresponds by (2.1) to a partition

$$P' = (x_1, \cdots, x_m) \tag{2.7}$$

of $(n-1)/2-(\epsilon_1-1)$. By (2.2) and (2.6), the largest summand of P' is $x_1=\epsilon_2$ (if $\kappa\geqq 2$; otherwise P' is empty). Since $\epsilon_1-1\geqq\epsilon_2$ by (2.3),

$$P = (\epsilon_1 - 1, x_1, \cdots, x_m) \tag{2.8}$$

is a partition of $(n-1)/2$ normalized by (1.3). Associating P with $[Q_1, Q_2]$, we obtain a correspondence between the diagonal partitions of n with odd summands and the partitions of $(n-1)/2$.

Actually, the results proved in this section are contained in a more general theorem of Sylvester [3, pp. 283–285], but it seems appropriate to reprove the special cases needed here, since Sylvester's paper is relatively unknown and suffers from an unduly complicated notation. It is to be noted, furthermore, that these correspondences give the results obtained in [1] by the use of generating functions where, however, the identity attributed to Durfee (p. 871) is really due to Euler [2, Theorem 351]. Also the numbers denoted there by d, e, h, evidently correspond to those denoted in this section by the same symbols.

We now write Δ for a δ-partition, i.e., a solution of (1.1) and (1.2). If S is the set (with multiplicities) of the summands of D or Δ, we write $D=D^S$, $\Delta=\Delta^S$. The results of this section show that Theorems 1 and 2 are an immediate consequence of the following.

LEMMA. *There is a* 1-1 *correspondence between the partitions* D^S *and* Δ^S *for any fixed set* S *of summands.*

This is a strengthened form of the lemma (not explicitly labeled) in [1, pp. 870–871], where it is proved that the two classes of partitions in question are equinumerous. For his sets A' and B' (p. 871) evidently correspond to our Q_1 and Q_2.

3. **Proof of the lemma.** (i) Given $D^S=[Q_1, Q_2]$, we determine the corresponding Δ^S as follows. Put

$$\alpha_{2f} = d_f, \qquad \alpha_{2g-1} = e_g, \qquad h + \kappa = \nu, \tag{3.1}$$

where d_f, e_g are given by (2.1). Thus the D^S correspond to the sequences $\{\alpha_t | 1 \leqq t \leqq \nu\}$, where the notation is self-explanatory. We then set

$$(3.2)\quad \begin{aligned} &\alpha_t = a_i \Leftrightarrow \alpha_t > \alpha_{t+1}; \qquad t = t_i \ (1 \leqq i \leqq r-1), \\ &\alpha_t = b_j \Leftrightarrow \alpha_t \leqq \alpha_{t+1}; \qquad t = u_j \leqq \nu - 1 \ (1 \leqq j \leqq s), \\ &a_\nu = a_r, \qquad r + s = \nu = t_r, \end{aligned}$$

in which $\{t_i | 1 \leqq i \leqq r\}$ and $\{u_j | 1 \leqq j \leqq s\}$ are increasing sequences. This gives

$$(3.3)\qquad \Delta^S = \sum_i a_i + \sum_j b_j.$$

We prove now that Δ^S satisfies (1.1) and (1.2). Recalling that Q_1 and Q_2 are partitions into distinct summands by (2.1), we see by (3.1) that $\alpha_t > \alpha_{t+2}$. Hence b_j and b_{j+1} are never consecutive terms of $\{\alpha_t\}$. Suppose $\alpha_t = a_i$, $i \leqq r-1$, so that $a_i > \alpha_{t+1}$. Then $a_{i+1} = \alpha_{t+1}$ or α_{t+2}, and $a_i > a_{i+1}$. If $\alpha_t = b_j$, then $b_{j+1} = \alpha_{t+2l} < \alpha_t$ or $b_{j+1} = \alpha_{t+2l-1} \leqq \alpha_{t+2l} < \alpha_t$. Thus $b_j > b_{j+1}$.

By the remark above, $u_j \geqq u_{j-1} + 2$. Since $u_1 \geqq 1$, this implies that $u_j \geqq 2j - 1$. Then by (3.2), $2j - 1 \leqq \nu - 1$, $j \leqq s \leqq \nu/2$ and $r + s = \nu$, so that $s \leqq r$. If $\alpha_t = b_j$, then $\alpha_{t+1} = a_i$ where $i = u_j + 1 - j \geqq j$. Hence $b_j \leqq a_i \leqq a_j$, and the proof is complete.

(ii) Given Δ^S, we obtain the corresponding D^S. Put $r + s = \nu$ and define

$$(3.4)\quad \begin{aligned} &i_s = \operatorname*{Max}_i\, (b_s \leqq a_i), \\ &i_j = \operatorname*{Max}_i\, (i < i_{j+1}) \ni b_j \leqq a_i, \qquad 1 \leqq j \leqq s - 1. \end{aligned}$$

Since $b_j \leqq a_j$ by hypothesis, the numbers i_j exist. Now form the sequence $\{\beta_v | 1 \leqq v \leqq \nu\}$, in which

$$(3.5)\quad \begin{aligned} &\beta_\nu = a_r; \qquad \beta_{v+1} = a_i,\ i = i_j \rightarrow \beta_v = b_j \rightarrow \beta_{v-1} = a_{i-1}, \\ &\beta_{v+1} = a_i,\ i \neq i_j \rightarrow \beta_v = a_{i-1}, \end{aligned}$$

where $v \leqq \nu - 1$ and a_{i-1} is absent for $i = 1$.

Now any sequence $\{\alpha_t\}$ which consists of the elements of S in some order and for which $\alpha_t > \alpha_{t+2}$ corresponds to a partition D^S by (3.1). We show that $\beta_v > \beta_{v+2}$ so that the sequences $\{\beta_v\}$ are included in the set of $\{\alpha_t\}$. Hence (3.5) and (3.1) give D^S.

Since the numbers i_j are distinct, b_j and b_{j+1} are not consecutive terms of $\{\beta_v\}$. Moreover $\{a_i\}$ and $\{b_j\}$ are strictly decreasing by (1.2). Suppose that $\beta_v = b_j$, then $\beta_{v+1} = a_i$ and $\beta_{v-1} = a_{i-1}$ for $i = i_j$. Now if $\beta_{\nu+2}=a_{i+1}$, $b_j>a_{i+1}$ by the maximal property of i_j. Suppose that $\beta_\nu=a_{i-2}$, $\beta_{\nu+2}=b_j$, so that $\beta_{\nu+1}=a_{i-1}$. Then $b_j\leqq a_i<a_{i-2}$. Hence $\beta_\nu>\beta_{\nu+2}$ in both cases. In the remaining case, β_ν and $\beta_{\nu+2}$ are both a's or both b's and the result is evident.

Now let U be the mapping defined by (3.4) and (3.5), and T that of (3.2), where T and U are evidently one-to-one. Since we have proved that Δ^S given by (3.3) satisfies the definition in §1, we can apply U to (3.3). It is easily seen that U is the inverse of T, so that the sequences $\{\alpha_t\}$ are included in the set of $\{\beta_\nu\}$. Hence the two sets are identical, for the converse has already been proved. We have therefore obtained the required 1-1 correspondence between the diagonal partitions D^S and the δ-partitions Δ^S, which proves the lemma.

For completeness, we indicate how a proof of the lemma can be obtained by applying more directly the method of [1]. As explained there, it suffices to consider the case where the elements of the summand set S are distinct. Hence we can assume that S is the set of the first k integers, so that $r+s=k$ in (1.1). Denote by $Q(k, m)$ (p. 870) the number of Δ^S for which $r-s\geqq m$. Thus $k-2s\geqq m$, and $0\leqq s\leqq\sigma=[(k-m)/2]$. Now we put $Q(k, m)=N_k(\sigma)$, the number of δ-partitions formed from the first k integers for which $s\leqq\sigma$. This definition leads to a simpler and more natural recurrence relation for this function than that given in the reference.

We have $b_s=1$ or $a_r=1$, where $b_s=1$ if $r=s$. If $\sigma<k/2$, we have $N_{k-1}(\sigma)$ δ-partitions with $a_r=1$, since $r>\sigma$ in this case, and $N_{k-1}(\sigma-1)$ of them with $b_s=1$, since removal of the 1 gives a δ-partition in both cases. Thus for $\sigma<k/2$ we have

$$N_k(\sigma) = N_{k-1}(\sigma) + N_{k-1}(\sigma - 1); \qquad k \geqq 2, \tag{3.6}$$

in which $\sigma\leqq(k-1)/2$ so that each term of (3.6) is defined. If $\sigma>k/2$, define $N_k(\sigma)=N_k(k-\sigma)$. Then (3.6) holds. The remaining case is when $\sigma=k/2$, in which the number of δ-partitions in question with $a_r=1$ is $N_{k-1}(\sigma-1)$, since $r>s$. Here $\sigma-1=(k-1)-\sigma$, so that $N_{k-1}(\sigma-1)=N_{k-1}(\sigma)$ by our definition. Now the number of Δ^S with $b_s=1$ is still $N_{k-1}(\sigma-1)$ and we conclude that (3.6) holds for all σ.

Since $N_k(0)=1$ $(k\geqq 1)$ and $N_1(1)=1$, (3.6) gives $N_k(\sigma)={}_kC_\sigma$. Now ${}_kC_\sigma$ is also the number of subsets of S with cardinality σ, i.e., the number of separations of S into two disjoint subsets A and B, where A has cardinality σ. Hence we seek a 1-1 correspondence between the δ-partitions Δ^S with at most σ summands in the second row and the ordered pairs $[A, B]$. This can be done using (3.6) and leads to a correspondence similar to that of this section.

4. Examples. The correspondence giving the direct proof of Theorems 1 and 2 is illustrated with two numerical examples. Since we have already illustrated in §2 the graphical dissection of a P_n into a D_n, the example illustrating Theorem 1 shows merely the correspondence between the partitions D_n and Δ_n given by the construction used in the lemma. For Theorem 2 it suffices, as we have seen, to consider the correspondence between the D_n and the $P_{[n/2]}$. Here we exemplify the most complicated case where n is odd and R_1 [see (2.4)] has $h-1$ summands, the details of the other cases being similar but simpler.

We use the same notation as above except that we denote a diagonal partition with odd summands by D°. All partitions or sequences occurring in a given example are in correspondence with each other, without explicit mention. For convenience, the partitions D and Δ are written on two lines with the summands d_f or b_j on the lower line and with parentheses or commas omitted. In the second example, reference numbers refer to the relevant definitions in §2.

(i) Given D, to obtain Δ (and conversely).

$$\text{a.}\quad D = \begin{matrix} 8\,7\,5\,2\,1 \\ 6\,5\,4\,3 \end{matrix},$$

$$\text{b.}\quad \{a_t\} = 8\,\mathbf{6}\,7\,\mathbf{5}\,5\,4\,\mathbf{2}\,3\,1,$$

where the b_j of (3.2) are boldfaced.

$$\text{c.}\quad \Delta = \begin{matrix} 8\,7\,5\,4\,3\,1 \\ 6\,5\,2 \end{matrix}.$$

To reverse this procedure we write Δ in the form

$$\begin{matrix} 8 & 7 & 5 & 4 & 3 & 1 \\ & 6 & 5 & & 2 & \end{matrix},$$

in which each b_j is aligned vertically with the corresponding a_i, $i=i_j$, of (3.4), and proceed as described there.

(ii) Given $P=P_{[n/2]}$, to obtain D°, n odd.

$$\text{a.}\quad P = (7\,6\,5\,5\,5\,1\,1) = (\epsilon_1 - 1, x_1, \cdots, x_6) \quad (2.8),$$

$$\text{b.}\quad P' = (6\,5\,5\,5\,1\,1) \quad (2.7),$$

$$\text{c.}\quad \begin{matrix} R_2 = 6\,4\,3\,2 \\ R_1 = 5\,2\,1 \end{matrix}, \qquad h = 4, \kappa = 5 \quad (2.4), (2.6),$$

$$\text{d.}\quad D^\circ = \begin{matrix} 15\,11\,7\,5\,3 \\ 9\,5\,3\,1 \end{matrix} \quad (2.3).$$

Here we have used (2.5).

I wish to thank the referee for some improvements in §2 of this paper.

References

1. B. Gordon, *Two new representations of the partition function*, Proc. Amer. Math. Soc. 13 (1962), 869–873.

2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, Oxford, 1960.

3. J. J. Sylvester, *A constructive theory of partitions*, Amer. J. Math. 5 (1882), 251–330.

Los Angeles, California

MULTIPLE TRANSITIVITY OF PRIMITIVE PERMUTATION GROUPS

L. A. MACHTINGER

Introduction. We wish to consider two theorems on permutation groups which are a generalization of a two-part theorem found in [1, pp. 66–67], a theorem concerned with a permutation group on a finite set. We shall remove the restriction of finiteness.

The following notation and definitions will be used in the discussion:

If Y is a set, then $|Y|$ denotes the cardinal number of Y. If g is a mapping of X into Z and $x \in X$, then xg denotes the image of x under g. If $Y \subseteq X$, then $(Y)g$ denotes the image of Y under g. If G is a group and $H \leqq G$ and $g \in G$, then $H^g = g^{-1}Hg$. A permutation group G on a set X is said to be r-ply (or r-fold) transitive on $Y \subseteq X$ (where r is a nonzero cardinal number and $|Y| \geqq r$), if (i) for every pair of subsets A and B of Y of cardinal number r and for any one-to-one map f of A onto B, there exists $g \in G$ such that $g|_A = f$ (where $g|_A$ is the restriction of g to A), and if (ii) $(Y)g = Y$ for every $g \in G$. A 1-fold transitive group on $Y \subseteq X$ is called transitive on Y.

A permutation group $G \neq 1$ on a set X is said to be imprimitive on $Y \subseteq X$ if: (i) Y can be written as a disjoint union of two or more nonvoid sets $\{S_\alpha\}_{\alpha \in \mathfrak{A}}$, (ii) for at least one $\alpha \in \mathfrak{A}$, $|S_\alpha| \geqq 2$ and (iii) given $g \in G$ and $\alpha \in \mathfrak{A}$, then $(S_\alpha)g = S_\beta$ for some $\beta \in \mathfrak{A}$. The sets S_α, $\alpha \in \mathfrak{A}$, are called imprimitivity sets of G on Y. If G is not imprimitive on Y, then G is called primitive on Y.

Received by the editors August 19, 1963.