A PARTICULAR CLASS OF LIE ALGEBRAS[1]

G. LEGER[2,3]

1. Introduction. Raffin announced in [2] that, under some additional conditions, nonassociative algebras over a field with

Property $*0$: $xy\in(x, y)$ (the linear space spanned by x and y), or

Property $*1$: $x(yz)\in(y, z)$ *for all* x, y, z, *in the algebra*

are Lie algebras. Accordingly, it seems of interest to know which Lie algebras have these properties. More generally, we shall consider Lie algebras L with

Property $*k$: $x_1(\cdots(x_k(uv))\cdots)\in(u, v)$ *for all* $x_1, \cdots, x_k, u, v$ *in* L,

and we shall prove the following

THEOREM. *Let L be a Lie algebra over any field F.*

(1) *If L has $*0$ then L is solvable.*

(2) *If L has $*k$ then either L is solvable or L is simple.*

(3) *If L is solvable with $*k$ then either L is nilpotent with $L^{k+2}=0$ or $L=(x)+A$ where A is a nonzero abelian ideal, $x\notin A$ and* $(\mathrm{ad}\, x)^{k+1}|_A=\alpha I$ *where I is the identity transformation of A and $\alpha\neq 0$ in F.*

(4) *If L is simple with $*k$ and if F has characteristic 0 then L has dimension 3.*

In the proof of part (3) of the theorem we need the following lemma which, in the characteristic 0 case, is an immediate consequence of the fact that $[L, L]\subset N$.

LEMMA 1. *Let L be a solvable Lie algebra with maximal nilpotent ideal N over any field F.*

(a) *If* $\mathrm{ad}\, L|_N$ *consists of nilpotent transformations, then L is nilpotent.*

(b) *If H is any nonzero ideal of L, then $H\cap N\neq 0$.*

(c) *If $x\in L$, $x\notin N$, then* $\mathrm{ad}\, x|_N\neq 0$.

We remark that $*0$ means that every subspace of L is a subalgebra. I will be used without further comment to denote the identity trans-

Received by the editors April 16, 1963 and, in revised form, May 31, 1963 and November 7, 1963.

[1] Research supported by National Science Foundation Grants NSF-G-14860 and NSF-G-784.

[2] The author wishes to thank the referee for suggestions which led to the final form of this paper and also W. Giles for this neat proof of part (a) of Lemma 1.

[3] Now at Tufts University.

formation of a vector space (just which space will be clear from the context) and we return to writing $[x, y]$ in place of xy.

2. Proof of the theorem.

LEMMA 2. *Let L be a Lie algebra with $*k$ over any field F and let V be a subspace of L. Let $y_1, \cdots, y_n$ be a basis of V, $x \notin V$ and $x_1, \cdots, x_k$ any elements of L. Then* $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k[x, y_j] = \alpha_j x + \beta_j y$ *with α_j, β_j in F and $\beta_1 = \beta_2 = \cdots = \beta_n$. If $[x, V] \subset V$, then either* $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k \operatorname{ad} x|_V = 0$ *or* $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k \operatorname{ad} x|_V = \alpha I$ *with $\alpha \neq 0$ in F. In the latter case,* $\operatorname{ad} x|_V$ *is a nonsingular transformation of V so that if each* $\operatorname{ad} x_j\colon V \to V$ *then* $\operatorname{ad} x_j|_V$ *is nonsingular on V for $j = 1, \cdots, k$.*

PROOF. Only the third sentence needs any comment since the rest follows at once from it. By $*k$, $\operatorname{ad} x_1 \cdots \operatorname{ad} x_k[x, y_i + y_j] = \alpha x + \beta(y_i + y_j) = (\alpha_i + \alpha_j)x + \beta_i y_i + \beta_j y_j$. If $i \neq j$, the linear independence of x, y_i, y_j forces $\beta_i = \beta = \beta_j$.

Proof of part (1) *of the theorem.* Since $*0$ is inherited by subalgebras we need only show that $[L, L] \neq L$ and we shall do this by induction on the dimension of L. If $\dim L = 1$ or 2, there is nothing to prove. If $\dim L > 2$, let y, v_1, v_2 be elements of L which are linearly independent over F. We write $[y, v_i] = \alpha_i y + \beta_i v_i$, $i = 1, 2$, and, by Lemma 2, $\beta_1 = \beta_2 = \tau$, say. Put $x = \alpha_2 v_1 - \alpha_1 v_2$ and we have that $[x, y] = -\tau x$. Now let M be a subspace of L such that $x \notin M$, $y \in M$, $(x) + M = L$, and apply Lemma 2 again ($\beta = 0$). Then we see that $[x, M] \subset (x)$. Thus $[L, L] = [(x) + M, (x) + M] \subset (x) + [M, M]$ so part (1) follows from the induction hypothesis applied to the subalgebra M.

Proof of part (2) *of the theorem.* Suppose that L is not solvable and has $*k$; we shall show that L is simple.

First we show that the center Z of L is zero. Since L is not solvable there exist $x_1, \cdots, x_k, u, v$ in L such that $t = \operatorname{ad} x_1 \cdots \operatorname{ad} x_k[u, v] \neq 0$. Now if $x \neq 0$ in Z then u, v, and x are linearly independent. Then $t = \alpha u + \beta v$ by $*k$ with α, β in F. Also $t = \operatorname{ad} x_1 \cdots \operatorname{ad} x_k[u, v + x] = \alpha_1 u + \beta_1(v + x)$ and $t = \operatorname{ad} x_1 \cdots \operatorname{ad} x_k[u + x, v] = \alpha_2(u + x) + \beta_2 v$ with the α_i, β_i in F. But now the linear independence of u, v and x forces $t = 0$ which is a contradiction. Thus Z is zero.

Next we show by induction on $\dim L$ that L has no nonzero abelian (and hence also no nonzero solvable) ideal. If $\dim L = 3$ then, since L is not solvable, L is simple and thus has no nontrivial ideals. If $\dim L > 3$, let J be an abelian ideal and suppose there exists $z \neq 0$ in J. Since the center of $L = 0$ we may choose x in L such that $[x, z] \neq 0$ and similarly (by induction) we may choose $x_k, \cdots, x_1$ in L such that ad $x_1 \cdots$ ad $x_k[x, z] \neq 0$. Then, by Lemma 2, we have ad $x|_J$ is nonsingular and hence, again by Lemma 2, $(\text{ad } x)^{k+1}|_J = \alpha I$ with $\alpha \neq 0$ in F. If $[y, J] \neq 0$ then, in the same way, ad $y|_J$ is nonsingular and again by Lemma 2, $(\text{ad } x)^k \text{ ad } y|_J = \beta I$ with $\beta \neq 0$ in F. Then $(\text{ad } x)^k \text{ ad}(\beta x - \alpha y)|_J = 0$ so, by the nonsingularity of $(\text{ad } x)^k|_J$, $\text{ad}(\beta x - \alpha y)|_J = 0$. Thus the map $x \to \text{ad } x|_J$ is a homomorphism of L onto a 1-dimensional Lie algebra and the kernel M of this map cannot be solvable (otherwise L would be), satisfies $*k$ and contains J. This leads to a contradiction since the induction hypothesis, applied to M, forces $J = 0$. Thus $J = 0$.

Now we show, again by induction on dim L, that L is simple. If dim $L = 3$ this is the case. Let S be any nonzero ideal of L. Then S cannot be solvable, by the last paragraph, and therefore S is simple by the induction hypothesis. Suppose $x \notin S$. Then $S^{k+2} \neq 0$ and we may choose $t \neq 0$ in S such that $t = \text{ad } s_1 \cdots \text{ad } s_k[u, v] \neq 0$ with $s_1, \cdots, s_k, u, v$ in S. Now we may argue (using x and t) just as in the first paragraph of this proof to conclude that $[x, S] \neq 0$. Now take w in S such that $[x, w] \neq 0$. Then, since the center of $S = 0$, we may choose $s_k', \cdots, s_i'$ in S inductively so that ad $s_i' \cdots$ ad s_k' $[x, w] \neq 0$. But now, by Lemma 2, ad $s_i'|_S$ is nonsingular which is absurd since s_i' is in S. Thus $L = S$ and hence is simple.

Proof of Lemma 1. (a) We proceed by induction on the dimension of L, the case dim $L = 0$ being trivial. If dim $L > 0$ and ad $L|_N = 0$ consists of nilpotent transformations, then Engel's theorem, applied to ad $L|_N$, shows that the center, Z, of L is not 0. Now note that (since ad $L|_N$ consists of nilpotent transformations) the maximal nilpotent ideal of L/Z is N/Z and thus L/Z also satisfies the condition of (a). Thus L/Z is nilpotent and therefore so is L.

(b) We proceed by induction on dim H, the case dim $H = 1$ being trivial. Suppose dim $H > 1$ then, since L is solvable, $[H, H] \neq H$. Now $[H, H]$ is an ideal of L and, if $[H, H] = 0$, then $H \subset N$, otherwise $[H, H] \cap N \neq 0$ by the induction hypothesis.

(c) Let $H = \{x \in L \mid [x, N] = 0\}$. H is an ideal of L and we claim $H \subset N$. If $H \not\subset N$ then, by (b), $A = H \cap N \neq 0$ and we let π denote the natural map of L onto L/A. Again, by (b), $\pi(H) \cap \overline{N} \neq 0$ where $\overline{N}$ is the maximal nilpotent ideal of L/A. Let $N_1 = \pi^{-1}(\pi(H) \cap \overline{N})$ and take $x \in N_1$. ad $\pi(x)$ is nilpotent on L/A and thus there exists k such that $(\text{ad } x)^k\{L\} \subset A \subset N$. But $x \in H$ so ad $x\{N\} = 0$, whence ad x is nilpotent on L. Thus $N_1 \subset N$ so that $N_1 \subset H \cap N$ and $\pi(N_1) = 0$ which is a contradiction.

Proof of part (3) *of the theorem.* We begin by noting that if L is nilpotent, then $*k$, applied to a regular basis of L, shows at once that $L^{k+2}=0$. If L is solvable but not nilpotent we let N denote the maximal nilpotent ideal of L and, using Lemma 1, take $x \notin N$ such that $\operatorname{ad} x|_N$ is not nilpotent. By Lemma 2, $\operatorname{ad} x|_N$ is a nonsingular transformation of N. Now we show that $[N, N]=0$. To see this, let U be a subspace of N such that $U+[N, N]=N$ and $U\cap[N, N]=0$. U generates N and we need only show that $[U, U]=0$. Take $u, v\in U$. Then $[u, v]\in[N, N]$ so that, by $*k$, $(\operatorname{ad} x)^k[u, v]\in U\cap[N, N]=0$. Since $\operatorname{ad} x|_N$ is nonsingular, $[u, v]=0$, i.e., $[U, U]=0$.

Next we show that N has co-dimension 1 in L. Let y be another element of L such that $y\notin N$. By Lemma 1, $\operatorname{ad} y|_N\neq 0$ and hence, since $\operatorname{ad} x|_N$ is nonsingular, $(\operatorname{ad} x)^k \operatorname{ad} y|_N\neq 0$. Now Lemma 2 shows that $(\operatorname{ad} x)^k \operatorname{ad} y|_N=\beta I$ for some $\beta\neq 0$ in F. Also, $(\operatorname{ad} x)^{k+1}|_N=\alpha I$ with $\alpha\neq 0$ in F. Thus $(\operatorname{ad} x)^k \operatorname{ad}(\beta x-\alpha y)|_N=0$. But $(\operatorname{ad} x)^k|_N$ is nonsingular and therefore $\operatorname{ad}(\beta x-\alpha y)|_N=0$. Thus, by Lemma 1, $\beta x-\alpha y\in N$.

Proof of part (4) *of the theorem.* Let L be simple and let F have characteristic 0. Take $x\neq 0$ in L such that $\operatorname{ad} x$ is semi-simple on L. Regarding L as an (x)-module, let M be an (x)-submodule of L such that $L=(x)+M$, $x\notin M$. $(\operatorname{ad} x)^{k+1}|_M\neq 0$, since $\operatorname{ad} x$ is semi-simple and therefore, by Lemma 2, $\operatorname{ad} x|_M$ is nonsingular. This implies that (x) is a maximal abelian subalgebra of L consisting of semi-simple elements and therefore [1, p. 219] is a Cartan subalgebra of L.

Now let K denote the algebraic closure of F and put $L^*=L\otimes_F K$. Then $\dim_F L=\dim_K L^*$. Further, L^* is semi-simple and $H=(x)\otimes_F K$ is a Cartan subalgebra of L^* [1, p. 224, Proposition 22]. Since $\dim_K H=1$, it follows first that L^* is simple and second, from the classification of simple Lie algebras, that $\dim_K L^*=3$.

References

1. C. Chevalley, *Theorie des groupes de Lie*, Tome II, Hermann, Paris, 1951.

2. R. Raffin, *Remarques sur certaines algèbres de Lie*, Symposium Internacional de Topología Algebraica, pp. 83–86, Universidad Nacional Autó. de Mexico and UNESCO, Mexico City, 1958.

Harvard University and
Western Reserve Univeresty