THE COEFFICIENTS IN AN ASYMPTOTIC EXPANSION

L. CARLITZ

Put

\[ e^z = \sum_{r=0}^{n} \frac{(nz)^r}{r!} + \frac{(nz)^n}{n!} S_n(z), \]

where \( n \) is a positive integer and \( z \) an arbitrary complex number. Ramanujan [4, p. 26] asserted (in a different notation) that

\[ S_n(1) = \frac{n!}{2} \left( \frac{e}{n} \right)^n - \frac{2}{3} + \frac{4}{135n} + O\left( \frac{1}{n^2} \right). \]

Copson [2] proved that \( \{S_n(-1)\} \) is a decreasing sequence with limit \(-1/2\) and derived an asymptotic series. In a recent paper, Buckholtz [1] proved that, for \( k \geq 1 \),

\[ S_n(z) = \sum_{r=0}^{k-1} \left( \frac{1}{n} \right)^r U_r(z) + O(n^{-k}) \]

uniformly in a certain region of the \( z \)-plane. The coefficients \( U_r(z) \) are determined by

(1) \[ U_r(z) = (-1)^r \frac{z d^r}{(1 - z)^2r+1} \frac{z}{1 - z}. \]

It follows from (1) that

(2) \[ U_r(z) = (-1)^r \frac{Q_r(z)}{(1 - z)^{2r+1}}, \]

where, for \( r \geq 1 \), \( Q_r(z) \) is a polynomial of degree \( r \) with positive integral coefficients.

To find an explicit expression for \( U_r(z) \), we put

(3) \[ F = F(z, t) = \sum_{k=0}^{\infty} U_k(z)^t/k!. \]

Then, by (1),

Received by the editors October 3, 1963.

1 This work was supported in part by NSF Grant GP-1593.
\[
\left( \frac{z}{1 - z} \frac{\partial}{\partial z} \right) F = - \sum_{k=0}^{\infty} U_{k+1}(z) \phi^k / k! = - \frac{\partial F}{\partial t},
\]

so that

\[D\]
\[
\frac{z}{1 - z} \frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0.
\]

The system

\[
\frac{1 - z}{z} \frac{dz}{dt} = \frac{dF}{dt} = 0
\]

has the particular integrals

\[
F, \quad ze^{-\tau t}.
\]

Hence (4) has the solution

\[E\]
\[
F = \phi(ze^{-\tau t}),
\]

where \(\phi\) is arbitrary. Since

\[
F(z, 0) = \frac{z}{1 - z},
\]

it is evident that

\[F\]
\[
\phi(ze^{-\tau}) = \frac{z}{1 - z}.
\]

Now it is known [3, p. 126, no. 214] that

\[
e^{-az}
\]
\[
\frac{e^{-az}}{1 - z} = \sum_{n=0}^{\infty} \frac{(n + a)^n}{n!} (ze^{-\tau})^n.
\]

It follows that

\[G\]
\[
\frac{z}{1 - z} = \sum_{n=1}^{\infty} \frac{n^n}{n!} (ze^{-\tau})^n.
\]

Comparing (7) with (6) it is clear that \(\phi\) is determined. Thus (5) becomes

\[H\]
\[
F(z, t) = \sum_{n=1}^{\infty} \frac{n^n}{n!} (ze^{-\tau t})^n.
\]

We have therefore
$$F(z, l) = \sum_{r=1}^{\infty} \frac{r^r}{r!} (ze^{-z})^r \sum_{k=0}^{\infty} \frac{(-1)^r r^k}{k!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{k^k}{k!} \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} (ze^{-z})^r,$$

so that

$$U_k(z) = (-1)^k \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} (ze^{-z})^r$$

$$= (-1)^k \sum_{r=1}^{\infty} \frac{r^{r+k}}{r!} z^r \sum_{s=0}^{\infty} (-1)^s \frac{r^s z^s}{s!}$$

$$= (-1)^k \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^{n+k}.$$

We put

$$S(n + k, n) = \frac{1}{n!} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^{n+k},$$

so that $S(n+k, n)$ is a Stirling number of the second kind [5, p. 33].

Thus

$$U_k(z) = (-1)^k \sum_{n=1}^{\infty} z^n S(n + k, n)$$

for all $k \geq 0$.

It follows from (2) and (10) that

$$Q_k(z) = (1 - z)^{2k+1} \sum_{n=1}^{\infty} z^n S(n + k, n).$$

If we put

$$Q_k(z) = \sum_{n=1}^{k} a_{kn} z^n \quad (k \geq 1),$$

it is clear that

$$a_{kn} = \sum_{j=0}^{n} (-1)^j \binom{2k+1}{j} S(n - j + k, n - j).$$

For example, since
we find that
\[ a(k) = 1 \quad (k \geq 1), \]
\[ a(k+1) = S(2 + k, 2) - (2k + 1)S(1 + k, 1) \]
\[ = \binom{2k+2}{2} - (2k + 1) \]
\[ = 2^{k+1} - 2(k + 1), \]
\[ a(k+2) = S(3 + k, 3) - (2k + 1)S(2 + k, 2) + \binom{2k + 1}{2}S(1 + k, 1) \]
\[ = \binom{3k+3}{2} - 3\cdot 2^{k+1} + 3 - (2k + 1)(2^{k+1} - 1) + \binom{2k + 1}{2} \]
\[ = 1\binom{3k+2}{2} - (2k + 3)2^{k+1} + (2k + 1)(k + 1). \]

In particular we have
\[ Q_1(z) = z, \quad Q_2(z) = z + 2z^2, \quad Q_3(z) = z + 8z^2 + 6z^3 \]
in agreement with Buckholtz.

It follows from (1) and (2) that
\[ Q_{k+1}(z) = (2k + 1)zQ_k(z) - z(z - 1)Q'_k(z). \]

Combining (13) with (11) we get the recurrence
\[ a(k, n) = na(k, n-1) + (2k - n)a(k-1, n-1), \]
from which it is clear that the \( a(k, n) \) are positive integers for \( 1 \leq n \leq k \).

By means of (14) we can easily compute the following table.

\[
\begin{array}{ccccccc}
1 & 1 & 2 \\
1 & 8 & 6 \\
1 & 22 & 58 & 24 \\
1 & 52 & 328 & 444 & 120 \\
1 & 114 & 1452 & 4400 & 3708 & 720 \\
\end{array}
\]

As a check we note that
\[ Q_k(1) = \sum_{n=1}^{k} a(k, n) = 1 \cdot 3 \cdot 5 \cdots (2k - 1); \]
this is an immediate consequence of (13).
ON CLASSES OF UNIVALENT CONTINUED FRACTIONS

T. L. HAYDEN1 AND E. P. MERKES2

1. Introduction. From results of Leighton and Scott [3], there is a unique one-to-one correspondence between formal power series \( w^{-1} + \sum_{n=2}^{\infty} c_n w^{-n} \) and C-fractions

\[
F(w) = \frac{1}{w} - \frac{a_1}{w^{\delta_1}} - \frac{a_2}{w^{\delta_2}} - \cdots - \frac{a_n}{w^{\delta_n}} - \cdots,
\]

where \( \delta_n \) is an integer, \( \delta_1 \geq 0, \delta_{n+1} + \delta_n \geq 1 \), and \( a_{n+p} = 0 \) whenever \( a_p = 0 \) for \( n = 1, 2, \ldots \). For a fixed continued fraction (1.1), let \( K_F \) denote the class of formal power series which correspond to C-fractions of the form

\[
F(w) = \frac{1}{w} - \frac{a'_1}{w^{\delta_1}} - \frac{a'_2}{w^{\delta_2}} - \cdots - \frac{a'_n}{w^{\delta_n}} - \cdots,
\]

where \( |a'_n| \leq |a_n|, n = 1, 2, \ldots \). In order that each power series in \( K_F \) represent an analytic function in \( |w| \geq 1 \) it is necessary and sufficient that \( |a_n| \leq g_n(1 - g_{n-1}) \), where \( 0 \leq g_{n-1} \leq 1, n = 1, 2, \ldots \), and \( g_{p-1} = 1 \) if and only if \( a_p = 0 \) [2, p. 374]. Conditions on the parameters \( g_n \) of the chain sequence \( \{g_n(1 - g_{n-1})\}_{n=1}^{\infty} \) which imply that each

Received by the editors October 16, 1963.

1 Sponsored by the Mathematics Research Center, U. S. Army, Madison, Wisconsin under Contract No. DA-11-022-ORD-2059.


License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use