MORE ABOUT INVERTIBLE OPERATORS WITHOUT ROOTS

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Introduction. It has been known for some time that on a complex infinite-dimensional Hilbert space there exist invertible operators without square roots, indeed without roots of any order, which therefore do not belong to the range of the exponential function. A first class of examples of such operators was described by Halmos, Lumer, and Schäffer [3]: the space considered was the separable Hilbert space of all complex-valued functions defined, analytic, and square-summable on a domain D of the complex plane (with the L^2-norm); the operator was the analytic position operator A defined by $(A\phi)(z) = z\phi(z)$, $z \in D$. It was shown that A is invertible and lacks a square root (indeed, a root of any order) if and only if D surrounds the origin but does not contain it. Halmos and Lumer [2] used the concept of multiplicity to show that the analytic position operator for such D is an interior point (in the norm topology for operators) of the set of invertible operators without roots.

Recently, Deckard and Pearcy [1] described another class of invertible square-root-less operators on a separable Hilbert space; the spectral properties of these operators are quite different from those of the operators discussed in [3] and [2]. We shall not be concerned with this class of examples in this paper.

Root-less operators are beginning to be used in various contexts: we mention, in particular, the disproof by Massera and Schäffer [5, p. 92], of an extension to Hilbert space of Floquet's Theorem on periodic differential equations. It appears that more information about such operators and more flexibility in their choice is desirable. To take the analytic position operator, for instance: there is of course some latitude in the choice of the domain D, but the description of the operator in terms of an orthonormal basis is awkward even in the simplest case, when D is an annulus centred at the origin, and quite unmanageable in other cases; besides, this description can not be "perturbed" or altered in any prescribed way as the application in hand might require. We remark, however, that the original definition has the advantage of making almost obvious the spectral structure that ensures the lack of roots.

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Another point deserves notice. In [5] the authors asserted, somewhat light-heartedly, that if A is the analytic position operator for an annulus, then "the proof in" [3] "may be modified to show that, indeed" A^h has no $2h$th root for any positive integer h. This is in fact true, but does not now seem quite so obvious to the present author as it did then. In [1], Deckard and Pearcy also remark, apropos of a square-root-less invertible operator B such that B^2 has no fourth root, "This shows that invertible operators can have roots of some order without having roots of all orders."

In this paper we first strengthen a particular case of the main result (Theorem 6) in [2] to show that operators of the kind described in the last paragraph can be made "to order," and actually constitute sets with interior points. We then describe, in terms of an orthonormal basis, a class of operators, sufficiently flexible for applications, that are invertible and root-less, and whose powers have roots of only the obvious orders. This class happens to contain the analytic position operator of every annulus centred at the origin.

Spectra and multiplicity. We recall a few pertinent concepts and properties mentioned in [3] and [2]. Let H be a complex Hilbert space, and let A be an operator on H, A^* the adjoint operator. We identify the complex number λ with the corresponding scalar operator. $\Pi(A)$ denotes the approximate point spectrum of A: a complex number λ does not belong to $\Pi(A)$ if and only if there exists a number $k>0$ such that $\|(A-\lambda)x\| \geqq k\|x\|$ for all $x \in H$. We shall require that part of the Spectral Mapping Theorem that asserts that, if p is any polynomial with complex coefficients, then $\Pi(p(A)) = p(\Pi(A))$, the latter set being the image of $\Pi(A)$ under the function p (cf. [3, p. 145]). Obviously $\Pi(S^{-1}AS) = \Pi(A)$ for any invertible operator S.

For any complex number λ, we define the *multiplicity* $m(A, \lambda)$ as the (orthogonal) dimension of the null-space of $A^* - \bar{\lambda}$. Obviously, $m(S^{-1}AS, \lambda) = m(A, \lambda)$ for any invertible operator S. We first require an algebraic result on multiplicities:

LEMMA 1. *If p is a nonconstant polynomial with complex coefficients and distinct roots $\lambda_1, \cdots, \lambda_h$, then $m(p(A), 0) = \sum_1^h m(A, \lambda_i)$.*

PROOF. We may assume without loss that $p(\zeta) = \prod_1^h (\zeta - \lambda_i)$. Let N_i be the $m(A, \lambda_i)$-dimensional null-space of $A^* - \bar{\lambda}_i$, $i = 1, \cdots, h$, and N the $m(p(A), 0)$-dimensional null-space of $(p(A))^* = \prod_1^h (A^* - \bar{\lambda}_i)$. We claim that N is the algebraic direct sum of the N_i, $i = 1, \cdots, h$; since the N_i and N are closed, this will yield the conclusion.

Consider any $y_i \in N_i$, $i=1, \cdots, h$. If $\sum_1^h y_i = 0$, repeated application of A^* yields $\sum_1^h \bar{\lambda}_i^j y_i = 0$, $j=0, \cdots, h-1$; since the $\bar{\lambda}_i$ are distinct, the nonvanishing of Vandermonde's determinant implies $y_i = 0$, $i=1, \cdots, h$. Thus the N_i are linearly independent.

The Lagrange interpolation formula yields the polynomial identity

$$1 = \sum_{j=1}^{h} \prod_{i \neq j} (\lambda_j - \lambda_i)^{-1}(\zeta - \lambda_i). \tag{1}$$

For any $x \in N$ we set $y_j = \prod_{i \neq j} (A^* - \bar{\lambda}_i)x$ and find $y_j \in N_j$, $j=1, \cdots, h$; (1) then implies $x = \sum_{j=1}^h (\prod_{i \neq j} (\bar{\lambda}_j - \bar{\lambda}_i))^{-1} y_j$. Therefore $N \subset \sum_1^h N_j$; the reverse inclusion follows from the obvious inclusions $N_j \subset N$, $j=1, \cdots, h$.

The next lemma summarizes some important results in [2].

LEMMA 2. *If K is a compact connected set of the complex plane and $K \cap \Pi(A) = \emptyset$, then $m(A, \lambda)$ has the same value $m(A, K)$ for all $\lambda \in K$, and there exists $\delta > 0$ such that $K \cap \Pi(B) = \emptyset$ and $m(B, K) = m(A, K)$ for every operator B with $\|B - A\| \leqq \delta$.*

PROOF. If $\Pi'(A)$ is the complement of $\Pi(A)$ in the complex plane, the function $A \to \Pi'(A)$ is lower semi-continuous (by [2, Theorem 1 and Lemma 1]; in the notation of that paper, $\Pi(A) = \Lambda_D(A)$, cf. p. 592); therefore $K \cap \Pi(B) = \emptyset$ if $\|B - A\|$ is sufficiently small. The conclusion about multiplicities follows from [2, Theorem 4], using the connectedness of K.

Powers and roots. We continue to consider an operator A on H. We denote by Γ the unit circumference in the complex plane; for any positive number ρ, $\rho\Gamma$ is then the circumference of centre 0 and radius ρ.

LEMMA 3. *Let A, a positive integer h, and a positive real number ρ be given. Then $\rho^h\Gamma \cap \Pi(A^h) = \emptyset$ if and only if $\rho\Gamma \cap \Pi(A) = \emptyset$; if this is the case, then $m(A^h, \rho^h\Gamma) = h \cdot m(A, \rho\Gamma)$.*

PROOF. Let p_h be the polynomial defined by $p_h(\zeta) = \zeta^h$. Then $\rho\Gamma$ is the complete inverse image of $\rho^h\Gamma$ under the mapping p_h, and therefore the above-mentioned Spectral Mapping Theorem for Π yields $\rho^h\Gamma \cap \Pi(A^h) = p_h(\rho\Gamma) \cap p_h(\Pi(A)) = p_h(\rho\Gamma \cap \Pi(A))$. This yields the first part of the statement. If $\rho^h\Gamma \cap \Pi(A^h) = \rho\Gamma \cap \Pi(A) = \emptyset$, Lemma 2 implies that $m(A, \rho\Gamma)$ and $m(A^h, \rho^h\Gamma)$ are well defined.

Denote by ξ a primitive hth root of 1, and let p be the polynomial $p(\zeta) = \zeta^h - \rho^h = \prod_1^h (\zeta - \xi^i\rho)$. Since the roots of p are distinct and lie on $\rho\Gamma$, Lemmas 1 and 2 imply $m(A^h, \rho^h\Gamma) = m(A^h, \rho^h) = m(p(A), 0) = \sum_1^h m(A, \xi^i\rho) = h \cdot m(A, \rho\Gamma)$.

THEOREM 1. *Let A be given, and assume that $\rho\Gamma\cap\Pi(A)=\varnothing$ for some positive real number ρ. There exists $\delta>0$ such that for every operator B with $\|B-A\|\leqq\delta$ the order of any root of B must divide $m(A,\rho\Gamma)$.*

PROOF. By Lemma 2 there exists δ such that if $\|B-A\|\leqq\delta$ then $\rho\Gamma\cap\Pi(B)=\varnothing$ and $m(B,\rho\Gamma)=m(A,\rho\Gamma)$. If $C^h=B$, Lemma 3 implies $\rho^{1/h}\Gamma\cap\Pi(C)=\varnothing$ and $m(A,\rho\Gamma)=m(B,\rho\Gamma)=h\cdot m(C,\rho^{1/h}\Gamma)$.

REMARK. Theorem 1 includes the trivial possibilities that $m(A,\rho\Gamma)=0$ (e.g., if $\rho>\|A\|$) or that $m(A,\rho\Gamma)$ is transfinite; in these cases, no restriction is implied for h.

A class of operators on l^2. We consider in particular the separable complex Hilbert space l^2 of square-summable sequences $x=(x_n)$, $n=\cdots,-1,0,1,\cdots$, of complex numbers, with the usual norm. We denote by V the unitary *shift operator* defined by $(Vx)_n=x_{n-1}$ for all n and every x.

For any sequence (Q_n), $n=\cdots,-1,0,1,\cdots$, of positive real numbers with $0<\inf_n Q_n\leqq\sup_n Q_n<\infty$ we define the operator Q on l^2 by $(Qx)_n=Q_nx_n$; we denote by M the class of all these "positive diagonal operators." All elements of M are positive Hermitian; M is a cone, and an abelian group under multiplication, with $(QR)_n=Q_nR_n$ and $(Q^{-1})_n=Q_n^{-1}$ for $Q,R\in M$. M is also invariant under transformation by V, and $(V^{-1}QV)_n=Q_{n+1}$ for each $Q\in M$.

We are particularly interested in the operators of the form VQ, with $Q\in M$ and satisfying

$$\limsup_{n\to-\infty} Q_n<\liminf_{n\to+\infty} Q_n. \tag{2}$$

We require a preliminary result about such Q.

LEMMA 4. *Assume that $Q\in M$ satisfies* (2) *and that the real number ρ lies strictly between the limits in* (2). *Then there exists a real number ϵ $0<\epsilon<1$, and $R\in M$ such that $S=\rho^{-1}(V^{-1}R^{-1}V)QR\in M$ satisfies*

$$S_n\leqq 1-\epsilon \quad \textit{for } n<0, \qquad S_n\geqq 1+\epsilon \quad \textit{for } n\geqq 0. \tag{3}$$

PROOF. We choose ϵ so small that

$$0<\limsup_{n\to-\infty} Q_n<\rho(1-\epsilon)<\rho(1+\epsilon)<\liminf_{n\to+\infty} Q_n<\infty. \tag{4}$$

We set $R_0=1$ and define (R_n) inductively by

$$R_{n+1}^{-1}R_n=\begin{cases}\min\{1,\rho(1-\epsilon)Q_n^{-1}\} & \text{for } n<0,\\ \max\{1,\rho(1+\epsilon)Q_n^{-1}\} & \text{for } n\geqq 0.\end{cases} \tag{5}$$

Since $R_n>0$ for all n, and the second member (5) is equal to 1 for all sufficiently large $|n|$, the sequence (R_n) defines an operator $R \in M$. Now $S_n = \rho^{-1} R_{n+1}^{-1} Q_n R_n$, so that (5) implies (3).

THEOREM 2. *Assume that $Q \in M$ satisfies* (2) *and that ρ lies strictly between the limits in* (2). *Then $\rho\Gamma \cap \Pi(VQ) = \emptyset$ and $m(VQ, \rho\Gamma) = 1$.*

PROOF. If ϵ, S are as in Lemma 4, we have $VS = \rho^{-1} V V^{-1} R^{-1} VQR = \rho^{-1} R^{-1}(VQ)R$; therefore $\Pi(VQ) = \rho\Pi(VS)$ and $m(VQ, \rho e^{i\theta}) = m(VS, e^{i\theta})$ for any real number θ. It is therefore sufficient to prove that $\Gamma \cap \Pi(VS) = \emptyset$ and (using Lemma 2) that $m(VS, 1) = m(VS, \Gamma) = 1$.

To establish the former assertion, we take an arbitrary real number θ and an arbitrary $x \in l^2$ and observe that $((VS - e^{i\theta})x)_n = S_{n-1}x_{n-1} - e^{i\theta}x_n$. We apply the triangle inequality to the "one-sided" sequences in the following computation:

$$\begin{aligned}
\|(VS - e^{i\theta})x\|^2 &\geqq \sum_{-\infty}^{0} (|x_n| - S_{n-1}|x_{n-1}|)^2 \\
&\quad + \sum_{1}^{\infty} (S_{n-1}|x_{n-1}| - |x_n|)^2 \\
&\geqq \left\{\left(\sum_{-\infty}^{0} |x_n|^2\right)^{1/2} - \left(\sum_{-\infty}^{-1} S_n^2 |x_n|^2\right)^{1/2}\right\}^2 \\
&\quad + \left\{\left(\sum_{0}^{\infty} S_n^2 |x_n|^2\right)^{1/2} - \left(\sum_{1}^{\infty} |x_n|^2\right)^{1/2}\right\}^2.
\end{aligned}$$

Now by (3), $\sum_{-\infty}^{-1} S_n^2 |x_n|^2 \leqq (1-\epsilon)^2 \sum_{-\infty}^{0} |x_n|^2$ and $\sum_0^\infty S_n^2 |x_n|^2 \geqq (1+\epsilon)^2 \sum_1^\infty |x_n|^2$. Therefore

$$\|(VS - e^{i\theta})x\|^2 \geqq \epsilon^2 \left(\sum_{-\infty}^{0} |x_n|^2 + \sum_{1}^{\infty} |x_n|^2\right) = \epsilon^2 \|x\|^2;$$

since this holds for each θ and all x, we have proved that $\Gamma \cap \Pi(VS) = \emptyset$.

Now $(((VS)^* - 1)x)_n = ((SV^{-1} - 1)x)_n = S_n x_{n+1} - x_n$. A vector x belongs to the null-space of $(VS)^* - 1$ if and only if

$$x_{n+1} = S_n^{-1} x_n, \qquad n = \cdots, -1, 0, 1, \cdots. \tag{6}$$

A sequence (x_n) that satisfies (6) is determined inductively once x_0, say, is given. By (3), it satisfies $|x_n| \leqq (1-\epsilon)^{-n}|x_0|$ for $n < 0$, $|x_n| \leqq (1+\epsilon)^{-n}|x_0|$ for $n \geqq 0$, so that $\sum_{-\infty}^{\infty} |x_n|^2 < \infty$, and $x \in l^2$ for any x_0. The null-space of $(VS)^* - 1$ is thus one-dimensional, and $m(VS, 1) = 1$, as was to be proved.

THEOREM 3. *Assume that $Q \in M$ satisfies* (2). *Then there exists $\delta > 0$ such that if A is any operator on l^2 with $\|A - VQ\| \leqq \delta$ and h is any positive integer, A^h has roots of each order that divides h, and of no other order.*

PROOF. Let ρ be as in Theorem 2. By Lemma 2 and Theorem 2, there exists $\delta > 0$ such that any A with $\|A - VQ\| \leqq \delta$ satisfies $\rho\Gamma \cap \Pi(A) = \varnothing$, $m(A, \rho\Gamma) = m(VQ, \rho\Gamma) = 1$. By Lemma 3, $\rho^h\Gamma \cap \Pi(A^h) = \varnothing$ and $m(A^h, \rho^h\Gamma) = h$. By Theorem 1, the order of any root of A^h must divide h. If k does divide h, $A^{h/k}$ is a kth root of A^h.

REMARK 1. By taking adjoints, we see that Theorem 3 remains valid if we replace VQ by $(VQ)^* = QV^{-1}$; if we then reverse the order of the sequences in l^2 we conclude that Theorem 3 remains valid as it stands if (2) is replaced by $\liminf_{n \to -\infty} Q_n > \limsup_{n \to +\infty} Q_n$. If we set $Q = 1 \in M$, however, $VQ = V$, being unitary, has a logarithm, and hence roots of any order.

REMARK 2. If we consider the *real* Hilbert space l^2_R of square-summable sequences of *real* numbers, V, Q are well defined on l^2_R (i.e., they are "real" operators on l^2); the norm of a "real" operator on l^2 is the same as its norm on l^2_R; and a "real" operator that has no "complex" roots of a certain order surely has no "real" ones either. It follows that Theorem 3 remains valid if l^2 is replaced by l^2_R.

REMARK 3. Let D be the annulus $\{z : \rho' < |z| < \rho''\}$. A complete orthonormal basis of the Hilbert space H_D of analytic functions mentioned in the Introduction is given by $\{\phi_m\}$, $m = \cdots, -1, 0, 1, \cdots$, where $\phi_m(z) = k_m z^{m-1}$, with $k_m = (2\pi \int_{\rho'}^{\rho''} t^{2m-1}\, dt)^{-1/2}$. If H_D is identified with l^2 by identifying the normalized Laurent series $\phi = \sum_{-\infty}^{\infty} x_n \phi_n$ with the sequence (x_n), the analytic position operator, which satisfies $A\phi = \sum_{-\infty}^{\infty} k_n^{-1} k_{n-1} x_{n-1} \phi_n$, becomes VQ, with $Q_n = k_n k_{n+1}^{-1}$. Now Schwarz's inequality yields $k_n^2 \geqq k_{n-1} k_{n+1}$, so that (Q_n) is strictly increasing, and it is easy to see that $0 < \lim_{n \to -\infty} Q_n = \rho' < \rho'' = \lim_{n \to +\infty} Q_n < \infty$, so that $Q \in M$ satisfies (2). Thus Theorem 3 is applicable: in particular, A^h has no roots of any but the obvious orders.

Root-less operators on other Hilbert spaces. Theorem 3 is a particular instance of a more general result (cf. [4]):

THEOREM 4. *On any infinite-dimensional real or complex Hilbert space there exists an invertible operator A_0 and $\delta > 0$ such that if A is any operator with $\|A - A_0\| < \delta$ and h is a positive integer, A^h has roots of each order that divides h, and of no other order.*

PROOF. Any complex infinite-dimensional Hilbert space is, up to an isometrical isomorphism, a direct sum $l^2 \oplus H$, where H is some nontrivial Hilbert space. We choose $Q \in M$ satisfying (2), and define A_0 as the direct sum $VQ \oplus 1$. A_0 is invertible, with $A_0^{-1} = Q^{-1}V^{-1} \oplus 1$. Let $\rho \neq 1$ be a positive real number that lies strictly between the limits in (2); we have $\Pi(A_0) = \Pi(VQ) \cup \Pi(1) = \Pi(VQ) \cup \{1\}$, so that $\rho\Gamma \cap \Pi(A_0) = \emptyset$ by Theorem 2. If N is the null-space of $(VQ)^* - \rho$, the null-space of $A_0^* - \rho$ is $N \oplus \{0\}$, since $1 - \rho$ is invertible on H; therefore $m(A_0, \rho\Gamma) = m(A_0, \rho) = m(VQ, \rho) = 1$, by Lemma 2 and Theorem 2. The conclusion follows as in Theorem 3.

In the real case the space is of the form $l_R^2 \oplus H_R$, and the operator $VQ \oplus 1$ is well defined. We may then consider $l_R^2 \oplus H_R$ as immersed in the complexification $l^2 \oplus H$ (where H is the complexification of H_R); the proof then follows from the proof in the complex case (cf. Remark 2 to Theorem 3).

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