

AN EXTENSION OF THE CONCEPT OF L_n SETS

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1. Introduction. A set S in the Euclidean plane E_2 is called an L_n set provided any two points of S can be joined by a polygonal line of at most n segments which lie entirely in S . A. M. Bruckner and J. B. Bruckner [1] have arrived at certain interesting results using L_n sets and the Hausdorff metric. We show here that very similar results may be obtained in the more general case of a complete, convex, locally compact metric space. As examples of spaces with these properties, we mention the Hausdorff metric space (see below) associated with such a space, the closed unit ball in the conjugate space (w^* -topology) of a separable normed linear space and, of course, all finite-dimensional Banach spaces.

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2. Definitions. Let (X, d) be any complete, convex, locally compact metric space. (A metric space is convex if for each pair of points x and y there is a point z such that $d(x, z) = d(z, y) = (1/2)d(x, y)$; such a point z is called a midpoint of x and y .) For any two points p and q in X , let m_1 be a midpoint of p and q , m_2 a midpoint of p and m_1 , m_3 a midpoint of m_1 and q , and so forth. We let $M(p, q) = \{m_i: i = 1, 2, \dots\}$, denote $\text{Cl}(M(p, q))$ by $\langle p, q \rangle$ and call $\langle p, q \rangle$ a line segment joining p and q . We note that this $\langle p, q \rangle$ may not be unique, but that in Euclidean space it is the usual line segment. The symbol $\langle p_0, p_1, \dots, p_n \rangle$ is used to denote any n -sided polygonal line joining p_0 to p_n , having consecutive intermediate vertices p_1, p_2, \dots, p_{n-1} . An L_n set is defined as in E_2 .

LEMMA 1. For (X, d) and $\langle p, q \rangle$ as defined above,

- (a) if x is a point of $\langle p, q \rangle$, then $d(p, x) + d(x, q) = d(p, q)$,
- (b) for each α such that $0 \leq \alpha \leq 1$, there is a unique point x of each $\langle p, q \rangle$ for which $d(p, x) = \alpha d(p, q)$,
- (c) each segment $\langle p, q \rangle$ is homeomorphic to the closed unit interval $[0, 1]$ of the real line, and hence is connected.

PROOF. (a) Lavish use of the triangle inequality in a simple induction proves the equality of this part for each consecutively chosen midpoint in $M(p, q)$. We may then extend the result to all of $\langle p, q \rangle$ by approximation with these midpoints.

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(b) By the method of construction, for each dyadic rational r in $[0, 1]$ there is an m_i in $M(p, q)$ such that $d(p, m_i) = rd(p, q)$. The dyadic rationals are dense in $[0, 1]$, so if α is in $[0, 1]$, we may find a sequence of dyadic rationals converging to α . The corresponding sequence of m_i 's is easily seen to converge to a point of $\langle p, q \rangle = \text{Cl}(M(p, q))$, and since the distance function is continuous, we have that $d(p, x) = \alpha d(p, q)$.

By construction, the points m_i are unique. By approximation with these midpoints we may extend the result to all of $\langle p, q \rangle$.

(c) By part (b), the function f defined by letting $f(x)$ be the point of $\langle p, q \rangle$ for which $d(p, f(x)) = xd(p, q)$ is well defined and one-to-one. We may easily show f to be continuous, then complete the proof by noting that any continuous, one-to-one function from a compact metric space to a metric space is a homeomorphism.

We now continue the definitions. For S any compact set in X , the ϵ -parallel body, $S_{(\epsilon)}$, of S is defined as

$$S_{(\epsilon)} = \{x \in X : d(x, S) \leq \epsilon\}.$$

We recall that the Hausdorff metric, ρ [4], may be defined on the family \mathcal{K} of all compact subsets of X by

$$\rho(S, T) = \inf\{\epsilon : S \subset T_{(\epsilon)} \text{ and } T \subset S_{(\epsilon)}\}.$$

3. Properties of (\mathcal{K}, ρ) . A. M. Macbeath [3] has examined the pair (\mathcal{K}, ρ) in some detail for the case $X = E_n$. Certain of the properties demonstrated are true for the situation under consideration, and those of utility are incorporated in the sequel.

THEOREM 1. *Let (X, d) be a complete, convex, locally compact metric space. Then*

- (a) *the pair (\mathcal{K}, ρ) is a metric space,*
- (b) *if S is a member of \mathcal{K} , then there exists an $\epsilon > 0$ such that $S_{(\epsilon)}$ is also a member of \mathcal{K} ,*
- (c) *every closed, bounded subset of X is compact.*

PROOF. (a) A well-known result [4].

(b) About each point of S there is a neighborhood whose closure is compact. From the collection of these neighborhoods we may extract a finite subcollection, U_1, U_2, \dots, U_n , which still covers S . We define functions f_i from X into the real numbers by $f_i(x) = d(x, X - U_i)$, then let $f(x) = \sum f_i(x)$; f is continuous. Now $s \in S$ implies that $f(s) > 0$, so there is a $\delta > 0$ such that $f(s) \geq \delta$ for every $s \in S$, and hence $f_i(s) \geq \delta/n$ for some i . Thus we see that $S_{(\delta/n)} \subset \cup U_i$, so $S_{(\delta/n)}$ is compact.

(c) Let p be a point of X . There is some $\epsilon > 0$ such that $\{p\}_{(\epsilon)}$ is

compact. Let $r = \sup \{ \epsilon : \{p\}_{(\epsilon)} \text{ is compact} \}$. Now suppose that r is finite. If $\{p\}_{(r)}$ is compact, we obtain an immediate contradiction using part (b). Hence $\{p\}_{(r)}$ is not compact, and thus for every η such that $r \geq \eta > 0$, $\{p\}_{(r-\eta)}$ is compact. Now let $\{x_n\}$ be a sequence in $\{p\}_{(r)}$. Let $S_1 = \{p\}_{(r-1/3)}$ if $r > 1/3$, and let $S_1 = \{p\}$ otherwise. Then for each n , either $x_n \in S_1$, or there is a point on some $\langle x_n, p \rangle$ whose distance from p is $\max(r-1/3, 0)$. In the latter case we call the point so found x'_n ; in the former we relabel x_n as x'_n . In either case, $d(x_n, x'_n) \leq 1/3$. Now in a metric space compactness is equivalent to sequential compactness [2], and $\{x'_n\} \subset S_1$, so we may find a convergent, hence Cauchy, subsequence $\{x'_{n(k)}\}$ of $\{x'_n\}$. Thus there exists a K such that $k \geq K$ implies $d(x'_{n(k)}, x'_{n(K)}) < 1/3$. Therefore, in the original sequence, $k \geq K$ implies

$$\begin{aligned} d(x_{n(k)}, x_{n(K)}) &\leq d(x_{n(k)}, x'_{n(k)}) + d(x'_{n(k)}, x'_{n(K)}) + d(x'_{n(K)}, x_{n(K)}) \\ &< 1/3 + 1/3 + 1/3 = 1. \end{aligned}$$

We denote $x_{n(K)}$ by ${}_1x_1$, $x_{n(K+1)}$ by ${}_1x_2$, etc.

Next, by considering the set $S_2 = \{p\}_{(r-1/6)}$ or $S_2 = \{p\}$, as above, relating $\{{}_1x_n\}$ to a sequence $\{{}_1x'_n\} \subset S_2$, and so forth, replacing $1/3$ by $1/6$ at every stage, we may find a subsequence, which we label $\{{}_2x_n\}$, of $\{{}_1x_n\}$ such that $d({}_2x_n, {}_2x_m) < 1/2$ for all m and n . At the q th stage this process yields a subsequence $\{{}_qx_n\}$ such that $d({}_qx_n, {}_qx_m) < 1/2^{q-1}$ for all m and n . The sequence $\{{}_nx_1\}$ of first terms of each successive subsequence has the property that $d({}_nx_1, {}_{n+1}x_1) < 1/2^{n-1}$ for all n . Hence $\{{}_nx_1\}$ is a Cauchy sequence and therefore converges in the complete metric space $\{p\}_{(r)}$. But this implies that $\{p\}_{(r)}$ is compact, a contradiction. Thus r cannot be finite, and this proves part (c).

The following theorem is proved for the case $X = E_n$ by A. M. Macbeath in [3]; in Theorem 1 we established all hypotheses necessary to that proof.

THEOREM 2. *If (X, d) is any complete, convex, locally compact metric space and (\mathcal{K}, ρ) is the associated Hausdorff metric space, then*

- (a) *if S^1, S^2, \dots is a decreasing sequence of members of \mathcal{K} , then $\lim S^k = \bigcap \{S^k : k = 1, 2, \dots\}$,*
- (b) *(\mathcal{K}, ρ) is complete.*

THEOREM 3. *Let n be a positive integer. The limit S of a sequence $\{S^k\}$ of compact L_n sets is a compact L_n set.*

PROOF. By Theorem 2(b), S is compact. In view of Theorem 1(c), it is easily seen that $\text{Cl}(\cup \{S^k : k = 1, 2, \dots\})$ is compact. Now let p and q be points of S , and let $\rho(S, S^k) = \epsilon_k$. Since $S^k \subset S_{(\epsilon_k)}$ and $\epsilon_k \rightarrow 0$

we may find sequences $\{p_k\}$ and $\{q_k\}$ such that p_k and q_k are in S^k and $\lim p_k = p$ and $\lim q_k = q$. There exist points $p_{1(k)}, \dots, p_{(n-1)(k)}$ in each S^k such that the n -line $\langle p_k, p_{1(k)}, \dots, q_k \rangle$ is contained in S^k . Recursively, we may choose a subsequence $\{S^j\}$ of $\{S^k\}$ for which the sequences $\{p_j\}, \{p_{1(j)}\}, \dots, \{q_j\}$ converge to the points p, p_1, \dots, q . Since S is closed and $\epsilon_j \rightarrow 0$, each p_i lies in S . Now to see that some $\langle p, p_1, \dots, p_{n-1}, q \rangle \subset S$, for each j let m_j be the midpoint of p_j and $p_{1(j)}$ lying on the n -line which was chosen in S^j . From the sequence $\{S^j\}$ we may extract a subsequence $\{S^{j(i)}\}$ for which the corresponding sequence $\{m_{j(i)}\}$ converges. Arguing as above, we see that $m = \lim m_{j(i)} \in S$. Further, since distance is preserved in the limit, m is a midpoint of p and p_1 . The same reasoning applies to all midpoints and dyadic points, so the closure of all these points, which is an n -line, joins p and q in S .

We have now produced sufficient machinery to effect the proofs of the following two lemmas. The author has been unable to improve on the proofs offered by A. M. and J. B. Bruckner, and they may be found in [1].

LEMMA 2. *If S is a compact, connected set and $\epsilon > 0$, then there exists a positive integer n such that $S_{(\epsilon)}$ is an L_n set.*

LEMMA 3. *The limit S of a sequence $\{S^k\}$ of compact, connected sets is compact and connected.*

LEMMA 4. *Any compact, connected set S is the limit of a sequence of compact L_n sets.*

PROOF. This follows from Theorem 1(b), Lemma 2 and the obvious fact that $\lim_{\epsilon \rightarrow 0} S_{(\epsilon)} = S$.

THEOREM 4. *A necessary and sufficient condition that a set S be compact and connected is that S be the limit of a sequence of compact L_n sets.*

PROOF. The necessity is stated in Lemma 4 and the sufficiency is an immediate consequence of Lemma 3, since by Lemma 1(c), L_n sets are connected.

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