STATIONARY MEASURES FOR BRANCHING PROCESSES

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Consider a Galton-Watson branching process of the type studied in [1]. The generating function of the family size distribution will be denoted by

$$f(s) = p_0 + p_1 s + p_2 s^2 + \cdots,$$

and we shall assume without further comment that $0 < p_0 < 1$. Harris [1, §11] has shown that the process admits a *stationary measure*, i.e., a set of non-negative numbers π_j $(j = 1, 2, \cdots)$ satisfying

$$\pi_j = \sum_{i=1}^{\infty} \pi_i P_{ij} \qquad (j = 1, 2, \cdots), \tag{1}$$

where P_{ij} is the probability that, if there are i individuals in one generation, there are j in the next. He has also shown that, if $\{\pi_j\}$ is any non-negative solution of (1), the generating function

$$\pi(s) = \sum_{j=1}^{\infty} \pi_j s^j \tag{2}$$

exists in $|s| < q$ (where q is the probability of extinction), and that, if $\{\pi_j\}$ is normalised so that

$$\pi(p_0) = 1, \tag{3}$$

then $\pi(s)$ satisfies Abel's functional equation

$$\pi[f(s)] = \pi(s) + 1 \qquad (|s| < q). \tag{4}$$

Conversely, if a solution $\pi(s)$ of (4) admits a power-series expansion (2) with non-negative coefficients, then the π_j form a stationary measure.

Harris has conjectured that the stationary measure for any Galton-Watson process is unique up to a constant multiplicative factor, or, equivalently, that there is exactly one stationary measure satisfying (3). The purpose of this note is to provide a counterexample to this conjecture.

Let ω be any entire function, not identically zero, which has period 1, and satisfies $\omega(0) = 0$. It follows from a well-known property of the equation (4) first noticed by Abel (cf. [1, §11.4]) that, if $\{\pi_j\}$ is a

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stationary measure, normalised by (3), and if a is any real number, then the function $\pi^a(s)$ given by

$$\pi^a(s) = \pi(s) + a\omega[\pi(s)]$$

satisfies (4). Moreover, $\omega[\pi(s)]$ admits a power-series expansion

$$\omega[\pi(s)] = \sum_{j=1}^{\infty} \chi_j s^j \tag{5}$$

in $|s| < q$. If it should happen that

$$\chi_j = O(\pi_j), \tag{6}$$

then, for all sufficiently small a, the coefficients $\pi_j + a\chi_j$ in the expansion of $\pi^a(s)$ are non-negative, and so form a normalised stationary measure which, for $a \neq 0$, is distinct from $\{\pi_j\}$. Thus, if (6) holds, then the stationary measure is not unique (even up to constant multiples).

Now consider the simple case

$$f(s) = q/(1 + q - s), \tag{7}$$

where $q < 1$ is the probability of extinction and the mean family size is $m = 1/q > 1$. For this process a normalised stationary measure is given by

$$\pi_j = (m^j - 1)/j \log m, \tag{8}$$

with

$$\pi(s) = \log\left(\frac{1 - s}{1 - ms}\right) \Big/ \log m. \tag{9}$$

Then

$$\chi_j = \frac{1}{2\pi i} \int \omega\left\{c \log\left(\frac{1 - s}{1 - ms}\right)\right\} \frac{ds}{s^{j+1}},$$

where $c = 1/\log m$ and the contour of integration goes once (anticlockwise) round the origin in $|s| < q$. The integrand is single-valued and regular in the closed complex plane cut along a straight line from q to 1. Moreover, since the imaginary part of $\log[(1-s)/(1-ms)]$ is bounded in this region, and since ω has a real period, the function

$$\omega\{c \log [(1 - s)/(1 - ms)]\}$$

is bounded. Hence we may deform the contour of integration into

one which goes from q to 1 above the cut and from 1 to q below it. Then

$$\chi_j = \frac{1}{2\pi i}\int_q^1 \Omega(x)\,\frac{dx}{x^{j+1}},$$

where

$$\begin{aligned}\Omega(x) = {} & \omega\{c \log [(1-x)/(mx-1)] - ic\pi\} \\ & - \omega\{c \log [(1-x)/(mx-1)] + ic\pi\}\end{aligned}$$

is bounded. Hence, if $|\Omega(x)| \leqq M$,

$$|\chi_j| \leqq \frac{M}{2\pi}\int_q^1 \frac{dx}{x^{j+1}} = \frac{M}{2\pi}\pi_j,$$

so that (6) is satisfied.

Thus, when $f(s)$ is given by (7), the stationary measure is not unique (even up to constant multiples). In fact, we can say more than this. Since, for any integer k, an admissible choice of $\omega(s)$ is $\sin(2\pi ks)$, it follows that the convex cone of stationary measures is infinite-dimensional.

A similar result can be proved for any generating function $f(s)$ of the "fractional linear" type in which the mean family size m is not equal to 1. This is in contrast to the case $m=1$, for which it is proved in [1] that the (normalised) stationary measure is unique.

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Reference

1. T. E. Harris, *The theory of branching processes*, Springer, Berlin, 1963.

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