# A CLASS OF SIMPLE LATTICE-ORDERED GROUPS

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A lattice-ordered group ($l$-group) is said to be *regular* if no positive element of the group is disjoint from any of its conjugates. It is well known that every simple regular $l$-group is totally ordered [5]. The subgroups of the reals are the most elementary examples of regular simple $l$-groups; other examples can be found in [2] and [6]. In this note we investigate a class of simple $l$-groups at the opposite extreme from the regular ones. We are concerned with $l$-groups which contain an *insular* (defined below) element. An insular element is, roughly speaking, an element which is strongly disjoint from one of its conjugates. In [4] it was shown that every $l$-group can be represented as an $l$-group of automorphisms of a totally ordered set, and it was shown that the $l$-group of automorphisms of the real line with bounded support is simple. It is natural to ask which simple $l$-groups can be represented as automorphisms of an ordered set with bounded support. Our main result is that these are exactly the simple $l$-groups containing an insular element. We also construct several examples of such groups.

If $L$ is a totally ordered set and $f$ is an order-preserving permutation of $L$, we call $f$ an *automorphism* of $L$. The *support* of $f$ consists of those $x \in L$ such that $xf \neq x$. An automorphism of $L$ is *bounded* if its support lies in a closed interval of $L$. An *$l$-group of automorphisms of* $L$ is a group of automorphisms of $L$ (under composition) which is a lattice under the operations $\cap$ and $\cup$ defined by $x(f \cap g) = (xf) \cap (xg)$ and dually. Such a group is a lattice-ordered group in the usual sense [1]. If $G$ is an $l$-group of automorphisms of $L$, $G$ is *o-primitive* on $L$ if there is no equivalence relation $E$ on $L$ such that (1) $E$ is a congruence; that is, for all $x, y \in L, f \in G$, $xEy$ implies $xfEyf$, and (2) $E$ is convex; that is, each $E$-class is a convex subset of $L$. For elements $f, g \geqq 1$ of an $l$-group $G, f$ is *right of* $g$ if for all $1 \leqq h \in G$, $g \cap h^{-1}fh = 1$. An element $g \in G$ is *insular* if for some conjugate $g^*$ of $g$, $g^*$ is right of $g$.

LEMMA. *Let $G$ be a transitive $l$-group of automorphisms of an ordered set $L$. An element $1 < g \in G$ is insular if and only if $g$ is bounded.*

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PROOF. Suppose the support of $g$ lies in the closed interval $[a, b]$ of $L$. By transitivity, there exists $1 \leqq h \in G$ such that $ah = b$. Let $g^* = h^{-1}gh$. Then for every $x \in [a, b]$, and for every $1 \leqq k \in G$, $xk^{-1}h^{-1} \leqq xh^{-1} \leqq bh^{-1} = a$. Hence $xk^{-1}h^{-1}g = xk^{-1}h^{-1}$. Thus $xk^{-1}g^*k = x$. Therefore the support of $k^{-1}g^*k$ lies outside the interval $[a, b]$, and so $g \cap k^{-1}g^*k = 1$.

Conversely, suppose $g^* = k^{-1}gk$ is right of $g$. Without loss of generality, $1 \leqq k$. There exists $x \in L$ such that $x < xg^*$. If there exists $y \in L$ such that $x < y < yg$, then by transitivity there exists $1 < f \in G$ such that $xf = y$, and thus $yf^{-1}g^*f = xg^*f > xf = y$, which implies $y(g \cap f^{-1}g^*f) > y$; that is, $g \cap f^{-1}g^*f > 1$, a contradiction. Hence the support of $g$ is bounded above by $x$. In a similar manner, there exists $z \in L$ such that $z < zg$. Let $w \leqq zk^{-1}$. Then $wk \leqq z$, so there exists $h \in G$ such that $1 \leqq h$ and $wkh = z$. Since $g \cap h^{-1}g^*h = 1$, $z = zh^{-1}k^{-1}gkh = wgkh \geqq wkh = z$. Hence $wgkh = wkh$, and $wg = w$. Therefore, the support of $g$ is bounded below by $zk^{-1}$.

THEOREM. *$G$ is a simple $l$-group containing an insular element if and only if $G$ is a transitive $o$-primitive $l$-group of bounded automorphisms of a totally ordered set.*

PROOF. Let $G$ be a simple $l$-group containing an insular element $g$. Every simple $l$-group is a transitive $l$-group of automorphisms of an ordered set [**4**, Theorem 3, Corollary 2]. By the lemma, $g$ must be bounded. It is easily seen that the bounded elements of $G$ form an $l$-ideal. Thus every element of $G$ is bounded. Hence $G$ is a transitive $l$-group of bounded automorphisms of an ordered set $L$. Of course, $G$ need not be $o$-primitive on $L$. Let the support of $1 \neq f \in G$ lie in the closed interval $[a, b]$ of $L$. If $E$ is any convex congruence on $L$, then

$$G_E = \{g \in G \mid xE(xg) \text{ for all } x \in L\}$$

is an $l$-ideal of $G$. Hence, for no proper convex congruence $E$ is $aEb$, since otherwise, $f \in G_E$ and hence $G = G_E$, which contradicts the transitivity of $G$. It follows that the union of any tower of proper convex congruences on $L$ is a proper convex congruence. By Zorn's lemma, there is a maximal proper convex congruence $M$ on $L$. The natural mapping induces a total order on $L' = L/M$. For $xM \in L'$ and $g \in G$, define $(xM)g = (xg)M$. Then $G$ is a transitive $o$-primitive $l$-group of bounded automorphisms of $L'$.

Conversely, let $G$ be a transitive $o$-primitive $l$-group of bounded automorphisms of an ordered set $L$. Let $\{1\} \neq N$ be an $l$-ideal of $G$. Define an equivalence relation $E$ on $L$ by: $xEy$ if and only if there exists $1 < f \in N$ such that $x \leqq yf$ and $y \leqq xf$. Then it is easily verified that $E$ is a convex congruence. Since $N \neq \{1\}$, and for any $f \in N$, $xE(xf)$ for all $x \in L$, at least one $E$-class contains more than one point. Therefore, since $G$ is $o$-primitive, there is just one $E$-class. Now let $1 < g \in G$. By assumption, the support of $g$ lies in some interval $[a, b]$. Since $aEb$, there exists $1 < f \in N$ such that $b \leqq af$. Hence, for any $x \in [a, b]$, $xg \leqq b \leqq af \leqq xf$. For any $x \in L \setminus [a, b]$, $xg = x \leqq xf$. Thus, $g \leqq f$, and as $N$ is convex, $g \in N$. Therefore, $G = N$, and $G$ is simple. Finally, by the lemma, every positive element of $G$ is insular. This completes the proof of the theorem.

COROLLARY 1. *If $G$ is a simple $l$-group with an insular element, then every positive element of $G$ is insular.*

COROLLARY 2. *If $G$ is a simple $l$-group with an insular element, then for every $1 < g \in G$ there is an infinite collection of pairwise disjoint conjugo'es of $g$.*

It is possible that the conclusion of Corollary 2 would follow from the weaker hypothesis that $G$ be simple and not totally ordered. It can be shown, using results in [3] that any simple nontotally ordered $l$-group contains an infinite collection of pairwise disjoint elements.

We close this note with some examples. Let $G$ be the $l$-group of all bounded automorphisms of the ordered set $L$. If $G$ is *o-doubly transitive* in the sense that for any $a, b, c, d \in L$ with $a < b$ and $c < d$, there exists $g \in G$ such that $ag = c$ and $bg = d$, then clearly $G$ is $o$-primitive on $L$. The following is also useful: *If $G$ is transitive on $L$ and $L$ is relatively complete, then $G$ is o-primitive on $L$.* For if $E$ were a nontrivial convex congruence on $L$, then there would be some nontrivial $E$-class containing an end point; but then $G$ could not be transitive on $L$.

In particular, if $L$ is an ordered field, then the group of automorphisms of $L$ is $o$-doubly transitive, and hence the $l$-group $G$ of bounded automorphisms of $L$ is simple. $G$ is also simple if $L$ is the long line, the inverted long line, or the double long line. A somewhat different example in which $L$ is relatively complete and yet not locally isomorphic to the reals arises in the following way. Consider the field $F$ of semi-infinite polynomials of the form $\sum_{i=-\infty}^{n} r_i x^i$ with integer exponents and real coefficients, ordered lexicographically from the largest exponent. Then the $l$-group of automorphisms of $F$ is $o$-doubly transitive. It follows that if $a_1 < a_2 < \cdots$ and $b_1 < b_2 < \cdots$ are bounded countable sequences of elements of $F$, then there is an automorphism of $F$ which maps $a_i$ onto $b_i$ for each $i$. A similar statement holds for decreasing sequences. Now let $L$ be the completion of $F$ by Dedekind cuts. Every automorphism of $F$ can be extended uniquely to an automorphism of $L$. Moreover, every element $a\in L$ is the limit of two sequences $\{a_i\}$ and $\{b_i\}$ of elements of $F$ such that

$$a_1 < a_2 < \cdots < a < \cdots < b_2 < b_1.$$

From this, it follows that the $l$-group $G$ of bounded automorphisms of $L$ is transitive on $L$. Hence $G$ is simple.

Finally, we give an example of a nontotally ordered simple $l$-group which does not contain an insular element. Let $t$ be that automorphism of the real line $R$ defined by $xt=x+1$. Let $G$ consist of all those automorphisms $f$ of $R$ such that $tf=ft$ ($f$ is "periodic"). Then $G$ is a transitive sub-$l$-group of the $l$-group of all automorphisms of $R$. To show that $G$ is simple, let $g$ and $f$ be positive elements of $G$. Then the support of $g$ meets the interval $[0, 1]$, where 0 and 1 denote real numbers throughout the following argument. Hence for a finite number of conjugates $g_1, g_2, \cdots, g_n$ of $g$, the support of $g^* = g_1 \cup g_2 \cup \cdots \cup g_n$ contains $[0, 1]$. Thus by periodicity $g^*$ has no fixed points in $R$. It follows that for some positive integer $m$, $1f < O(g^*)^m$. Therefore, for all $x \in [0, 1]$,

$$xf \leqq 1f < O(g^*)^m \leqq x(g^*)^m.$$

By periodicity again, $yf < y(g^*)^m$ for all $y \in R$. That is, $f < (g^*)^m$. Thus any $l$-ideal of $G$ containing $g$ also contains $f$. Therefore, $G$ is simple. Clearly, $G$ does not contain an element of bounded support, and therefore, no element of $G$ is insular.

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