A CLASS OF SIMPLE LATTICE-ORDERED GROUPS

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A lattice-ordered group (l-group) is said to be regular if no positive element of the group is disjoint from any of its conjugates. It is well known that every simple regular l-group is totally ordered [5]. The subgroups of the reals are the most elementary examples of regular simple l-groups; other examples can be found in [2] and [6]. In this note we investigate a class of simple l-groups at the opposite extreme from the regular ones. We are concerned with l-groups which contain an insular (defined below) element. An insular element is, roughly speaking, an element which is strongly disjoint from one of its conjugates. In [4] it was shown that every l-group can be represented as an l-group of automorphisms of a totally ordered set, and it was shown that the l-group of automorphisms of the real line with bounded support is simple. It is natural to ask which simple l-groups can be represented as automorphisms of an ordered set with bounded support. Our main result is that these are exactly the simple l-groups containing an insular element. We also construct several examples of such groups.

If L is a totally ordered set and f is an order-preserving permutation of L, we call f an automorphism of L. The support of f consists of those x ∈ L such that xf ≠ x. An automorphism of L is bounded if its support lies in a closed interval of L. An l-group of automorphisms of L is a group of automorphisms of L (under composition) which is a lattice under the operations \( \land \) and \( \lor \) defined by \( x(f \land g) = (xf) \land (xg) \) and dually. Such a group is a lattice-ordered group in the usual sense [1]. If G is an l-group of automorphisms of L, G is o-primitive on L if there is no equivalence relation E on L such that (1) E is a congruence; that is, for all x, y ∈ L, f ∈ G, xEy implies xfEyf, and (2) E is convex; that is, each E-class is a convex subset of L. For elements f, g ≥ 1 of an l-group G, f is right of g if for all 1 ≤ h ∈ G, g \( \land h^{-1}fh = 1 \). An element g ∈ G is insular if for some conjugate g* of g, g* is right of g.

Lemma. Let G be a transitive l-group of automorphisms of an ordered set L. An element 1 < g ∈ G is insular if and only if g is bounded.

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Proof. Suppose the support of $g$ lies in the closed interval $[a, b]$ of $L$. By transitivity, there exists $1 \leq h \in G$ such that $ah = b$. Let $g^* = h^{-1}gh$. Then for every $x \in [a, b]$, and for every $1 \leq k \in G$, $xk^{-1}h^{-1} \leq xh^{-1} \leq bh^{-1} = a$. Hence $xk^{-1}h^{-1}g = xk^{-1}h^{-1}$. Thus $xk^{-1}g^*k = x$. Therefore the support of $k^{-1}g^*k$ lies outside the interval $[a, b]$, and so $g \cap k^{-1}g^*k = 1$.

Conversely, suppose $g^* = k^{-1}gk$ is right of $g$. Without loss of generality, $1 \leq k$. There exists $x \in L$ such that $x < xg^*$. If there exists $y \in L$ such that $x < y < yg$, then by transitivity there exists $1 < f \in G$ such that $xf = y$, and thus $yf^{-1}g^*f = xg^*f > xf = y$, which implies $y(\cap f^{-1}g^*f) > y$; that is, $g \cap f^{-1}g^*f > 1$, a contradiction. Hence the support of $g$ is bounded above by $x$. In a similar manner, there exists $z \in L$ such that $z < zg$. Let $w \leq zk^{-1}$. Then $wk \leq z$, so there exists $h \in G$ such that $1 \leq h$ and $whk = z$. Since $g \cap h^{-1}g^*h = 1$, $z = zh^{-1}k^{-1}gkh = wgkh \geq whk = z$. Hence $wgkh = whk$, and $wg = w$. Therefore, the support of $g$ is bounded below by $zk^{-1}$.

Theorem. $G$ is a simple $l$-group containing an insular element if and only if $G$ is a transitive $o$-primitive $l$-group of bounded automorphisms of a totally ordered set.

Proof. Let $G$ be a simple $l$-group containing an insular element $g$. Every simple $l$-group is a transitive $l$-group of automorphisms of an ordered set [4, Theorem 3, Corollary 2]. By the lemma, $g$ must be bounded. It is easily seen that the bounded elements of $G$ form an $l$-ideal. Thus every element of $G$ is bounded. Hence $G$ is a transitive $l$-group of bounded automorphisms of an ordered set $L$. Of course, $G$ need not be $o$-primitive on $L$. Let the support of $1 \neq f \in G$ lie in the closed interval $[a, b]$ of $L$. If $E$ is any convex congruence on $L$, then

$$G_E = \{g \in G \mid xE(xg) \text{ for all } x \in L\}$$

is an $l$-ideal of $G$. Hence, for no proper convex congruence $E$ is $aEb$, since otherwise, $f \in G_E$ and hence $G = G_E$, which contradicts the transitivity of $G$. It follows that the union of any tower of proper convex congruences on $L$ is a proper convex congruence. By Zorn’s lemma, there is a maximal proper convex congruence $M$ on $L$. The natural mapping induces a total order on $L' = L/M$. For $xM \in L'$ and $g \in G$, define $(xM)g = (xg)M$. Then $G$ is a transitive $o$-primitive $l$-group of bounded automorphisms of $L'$.

Conversely, let $G$ be a transitive $o$-primitive $l$-group of bounded automorphisms of an ordered set $L$. Let $\{1\} \neq N$ be an $l$-ideal of $G$. Define an equivalence relation $E$ on $L$ by: $xEy$ if and only if there exists $1 < f \in N$ such that $x \leq yf$ and $y \leq xf$. Then it is easily verified
that $E$ is a convex congruence. Since $N \neq \{1\}$, and for any $f \in N$, $x E(x f)$ for all $x \in L$, at least one $E$-class contains more than one point. Therefore, since $G$ is $o$-primitive, there is just one $E$-class. Now let $1 < g \in G$. By assumption, the support of $g$ lies in some interval $[a, b]$. Since $a \in b$, there exists $1 < f \in N$ such that $b \leq af$. Hence, for any $x \in [a, b]$, $xg \leq b \leq af \leq xf$. For any $x \in L \setminus [a, b]$, $xg = x \leq xf$. Thus, $g \leq f$, and as $N$ is convex, $g \in N$. Therefore, $G = N$, and $G$ is simple. Finally, by the lemma, every positive element of $G$ is insular. This completes the proof of the theorem.

**Corollary 1.** If $G$ is a simple $l$-group with an insular element, then every positive element of $G$ is insular.

**Corollary 2.** If $G$ is a simple $l$-group with an insular element, then for every $1 < g \in G$ there is an infinite collection of pairwise disjoint conjugates of $g$.

It is possible that the conclusion of Corollary 2 would follow from the weaker hypothesis that $G$ be simple and not totally ordered. It can be shown, using results in [3] that any simple nontotally ordered $l$-group contains an infinite collection of pairwise disjoint elements.

We close this note with some examples. Let $G$ be the $l$-group of all bounded automorphisms of the ordered set $L$. If $G$ is $o$-doubly transitive in the sense that for any $a, b, c, d \in L$ with $a < b$ and $c < d$, there exists $g \in G$ such that $ag = c$ and $bg = d$, then clearly $G$ is $o$-primitive on $L$. The following is also useful: If $G$ is transitive on $L$ and $L$ is relatively complete, then $G$ is $o$-primitive on $L$. For if $E$ were a nontrivial convex congruence on $L$, then there would be some nontrivial $E$-class containing an end point; but then $G$ could not be transitive on $L$.

In particular, if $L$ is an ordered field, then the group of automorphisms of $L$ is $o$-doubly transitive, and hence the $l$-group $G$ of bounded automorphisms of $L$ is simple. $G$ is also simple if $L$ is the long line, the inverted long line, or the double long line. A somewhat different example in which $L$ is relatively complete and yet not locally isomorphic to the reals arises in the following way. Consider the field $F$ of semi-infinite polynomials of the form $\sum_{i=0}^{\infty} r_{i}x^{i}$ with integer exponents and real coefficients, ordered lexicographically from the largest exponent. Then the $l$-group of automorphisms of $F$ is $o$-doubly transitive. It follows that if $a_{1} < a_{2} < \cdots$ and $b_{1} < b_{2} < \cdots$ are bounded countable sequences of elements of $F$, then there is an automorphism of $F$ which maps $a_{i}$ onto $b_{i}$ for each $i$. A similar statement holds for decreasing sequences. Now let $L$ be the completion of $F$ by Dedekind cuts. Every automorphism of $F$ can be extended uniquely
to an automorphism of \( L \). Moreover, every element \( a \in L \) is the limit of two sequences \( \{ a_i \} \) and \( \{ b_i \} \) of elements of \( F \) such that

\[
a_1 < a_2 < \cdots < a < \cdots < b_2 < b_1.
\]

From this, it follows that the \( l \)-group \( G \) of bounded automorphisms of \( L \) is transitive on \( L \). Hence \( G \) is simple.

Finally, we give an example of a nontotally ordered simple \( l \)-group which does not contain an insular element. Let \( t \) be that automorphism of the real line \( R \) defined by \( xt = x + 1 \). Let \( G \) consist of all those automorphisms \( f \) of \( R \) such that \( tf = ft \) (\( f \) is “periodic”). Then \( G \) is a transitive sub-\( l \)-group of the \( l \)-group of all automorphisms of \( R \). To show that \( G \) is simple, let \( g \) and \( b \) be positive elements of \( G \). Then the support of \( g \) meets the interval \([0, 1] \), where 0 and 1 denote real numbers throughout the following argument. Hence for a finite number of conjugates \( g_1, g_2, \ldots, g_n \) of \( g \), the support of \( g^* = g_1 \cup g_2 \cup \cdots \cup g_n \) contains \([0, 1] \). Thus by periodicity \( g^* \) has no fixed points in \( R \). It follows that for some positive integer \( m \), \( 1f < O(g^*)^m \). Therefore, for all \( x \in [0, 1] \),

\[
x f \leq 1f < O(g^*)^m \leq x(g^*)^m.
\]

By periodicity again, \( y f < y(g^*)^m \) for all \( y \in R \). That is, \( f < (g^*)^m \). Thus any \( l \)-ideal of \( G \) containing \( g \) also contains \( f \). Therefore, \( G \) is simple.

Clearly, \( G \) does not contain an element of bounded support, and therefore, no element of \( G \) is insular.

References


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