ON THE ZEROS OF A CONFLUENT HYPERGEOMETRIC FUNCTION[1]

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1. Introduction. In this note, by using an integral representation due to Erdélyi, we prove a theorem concerning the zeros of the confluent hypergeometric function $\Psi(a, c|z)$. As a corollary, we prove that the zeros of the Bessel polynomials for $\delta \geqq 0$, $n \geqq 1$, lie in the left-half plane.

2. Fundamental theorems.

THEOREM. *Let a and c be real and $2a-c>-1$. Then any zero z_ν of $\Psi(a, c|z)$ must satisfy* $\mathrm{Re}(z_\nu)<0$.

PROOF. In the formula [2, p. 287, equation (21)]

$$\Gamma(\gamma)\Psi(a, c \mid x)\Psi(a', c \mid y)$$

$$= \int_0^\infty e^{-t}t^{\gamma-1}(x+t)^{-a}(y+t)^{-a'}{}_2F_1\left(\begin{matrix} a, a' \\ \gamma \end{matrix} \middle| \frac{t(x+y+t)}{(x+t)(y+t)}\right) dt, \tag{1}$$

$$\gamma = a + a' - c + 1, \qquad \mathrm{Re}(\gamma) > 0,\ xy \neq 0,$$

let $x=\sigma+i\tau=z$, $y=\sigma-i\tau=\bar{z}$, $a=a'$, $\gamma=2a-c+1$. Then

$$\Psi(a, c \mid z)\Psi(a, c \mid \bar{z})$$

$$= \int_0^\infty \frac{e^{-t}t^{\gamma-1}[(t+\sigma)^2+\tau^2]^{-a}}{\Gamma(\gamma)}\, {}_2F_1\left(\begin{matrix} a, a \\ \gamma \end{matrix} \middle| \frac{t(t+2\sigma)}{(t+\sigma)^2+\tau^2}\right) dt. \tag{2}$$

Since $\gamma>0$, the integrand of (2) is a series of strictly positive terms. Thus we have

$$\Psi(a, c \mid z)\Psi(a, c \mid \bar{z}) > 0, \qquad \mathrm{Re}(z) \geqq 0,\ z \neq 0. \tag{3}$$

A known result [6] excludes the point $z=0$ from consideration, and the proof of the theorem is complete.

We now define the Bessel polynomials by

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$$P_n^{(\delta)}(z) = \sum_{k=0}^{n} \binom{n}{k} (n+\delta)_k z^{n-k}, \qquad \delta \geqq 0,\ n \geqq 1, \tag{4}$$

where

$$\begin{aligned} (n+\delta)_k &= (n+\delta)(n+\delta+1)\cdots(n+\delta+k-1), \\ (n+\delta)_0 &= 1. \end{aligned} \tag{5}$$

For a treatment of these polynomials and specializations, see, for example, [1], [3], [4].

McCarthy [5] has proved that for n fixed and δ *sufficiently large*, the zeros of (4) are in the left-half plane. Applying our theorem, we may prove a more general result.

COROLLARY. *Every zero* $z_\nu^{(n)}$ *of* (4) *satisfies* $\mathrm{Re}(z_\nu^{(n)}) < 0$.

PROOF. Consider the function

$$f(z) = \Psi(-n, -2n-\delta+1 \mid z). \tag{6}$$

We also have

$$f(z) = z^{2n+\delta}\Psi(n+\delta, 2n+\delta+1 \mid z). \tag{7}$$

We distinguish two cases.[2] First let δ be nonintegral. Then by [2, p. 257, equation (7)] and (6),

$$\begin{aligned} f(z) &= (n+\delta) \sum_{k=0}^{n} \binom{n}{k} \frac{(-z)^k}{(-2n-\delta+1)_k} \\ &= P_n^{(\delta)}(z), \end{aligned} \tag{8}$$

as can be seen by turning the sum around. Now let δ be integral and use (7) and [2, p. 261, equation (13)].

$$\begin{aligned} f(z) &= (n+\delta)_n \sum_{k=0}^{2n+\delta-1} \binom{n}{k} \frac{(-z)^k}{(-2n-\delta+1)_k}, \\ &= P_n^{(\delta)}(z). \end{aligned} \tag{9}$$

Our theorem furnishes the desired result for $\delta > 0$, with $a = -n$, $c = -2n-\delta+1$, and $\gamma = \delta$. Also, we may let $\delta \to 0$ and the integral (2) in the limit exists, because

[2] This is necessary because $\Psi(a, c \mid x)$ assumes two different forms depending on whether or not c is an integer. Note also (4) defines a polynomial of degree n, $n \geqq 0$, $\delta \neq -1, -2, \cdots$, but we prefer to define $P_n^{(\delta)}(z)$ so the corollary holds.

$$
\begin{aligned}
f(z)f(\bar{z}) &= \lim_{\delta\to 0} \int_0^\infty \frac{e^{-t}t^{\delta-1}}{\Gamma(\delta)} [(t+\sigma)^2+\tau^2]^n\, dt \\
&\quad + \int_0^\infty e^{-t}[(t+\sigma)^2+\tau^2]^n \sum_{k=1}^{n} \frac{t^{k-1}(-n)_k^2}{k!\Gamma(k)} \\
&\quad \cdot \left[\frac{(t+2\sigma)}{(t+\sigma)^2+\tau^2}\right]^k dt \\
&= (\sigma^2+\tau^2)^n + \int_0^\infty e^{-t}[(t+\sigma)^2+\tau^2]^n \sum_{k=1}^{n} \frac{t^{k-1}(-n)_k^2}{k!\Gamma(k)} \\
&\quad \cdot \left[\frac{(t+2\sigma)}{(t+\sigma)^2+\tau^2}\right]^k dt.
\end{aligned} \tag{10}
$$

Thus the same argument applies in the limiting case and the proof of the corollary is complete.

Added in proof. We have noted that Buchholz (*Die konfluente hypergeometrische Funktion*, p. 90, Springer, Berlin, 1953) proves a similar theorem (excluding $z=0$) for the related function $W_{k,m}(z)$.

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