A NEW PROOF OF DEICKE’S THEOREM ON HOMOGENEOUS FUNCTIONS

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We denote by $R_n$ the $n$-dimensional number space of points \{ $x^1, x^2, \cdots, x^n$ \}, where the $x^i$ are real numbers, and we use $R'_n$ to denote $R_n$ with the point \{ $0, 0, \cdots, 0$ \} removed. Let $L$ be a positive function of class $C^4$ defined on $R'_n$ and positively homogeneous of degree one. Then, introducing the matrix $g$ of elements

$$g_{ij} = \frac{\partial^2(\frac{1}{2}L)}{\partial x^i \partial x^j},$$

we give a new proof of the following theorem, due originally to A. Deicke [1].

Theorem. Let $\det g$ be constant on $R'_n$. Then $g$ is constant on $R'_n$.

It is known that the assumptions made imply that the matrix $g$ is positive definite [1]. We first prove

Lemma 1. Let $x, y$ be any two points in $R'_n$. Then $\operatorname{Tr} g^{-1}(x) g(y) \geq n$.

Proof. Since the matrices $g(x), g(y)$ are positive definite, the characteristic roots of $g(y)$ with respect to $g(x)$ are all positive. These roots are also the characteristic roots of the matrix $g^{-1}(x) g(y)$ so that, using the inequality between arithmetic and geometric means,

$$\operatorname{Tr} g^{-1}(x) g(y) \geq n(\det g^{-1}(x) g(y))^{1/n} = n.$$

We next introduce the elliptic differential operator

$$\Delta = \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j},$$

where $g^{ij}$ denotes the general element of the matrix $g^{-1}(x)$ and prove

Lemma 2. The matrix $\Delta g$ is positive semi-definite.

Proof. Define a function $\phi_x$ by $\phi_x(y) = \operatorname{Tr} g^{-1}(x) g(y)$. Since $\phi_x(x) = n$, Lemma 1 shows that $\phi_x$ has a minimum at $y = x$ and hence the matrix of elements

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\frac{\partial^2 \phi_x}{\partial y^a \partial y^b}
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is positive semi-definite for \(y = x\). This matrix is also equal to \(\Delta g\) for \(y = x\).

We complete the proof of the theorem by using a theorem due to E. Hopf [2, Theorem 2.1]. Lemma 2 implies that, for each \(h\), \(\Delta g_{hh} \geq 0\). Since \(g_{hh}\) is positively homogeneous of degree zero and hence attains a maximum on \(R_n\), Hopf's theorem shows that \(g_{hh}\) is constant on \(R_n\). Lemma 2 now implies that \(\Delta g_{hh} = 0\) for all \(h, k\) and, as before, Hopf's theorem shows that \(g_{hh}\) is constant on \(R_n\).

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REFERENCES

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