THE MAXIMUM TERM OF AN ENTIRE SERIES WITH GAPS

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Let \( \sum a_n z^n \) denote the power series for an entire function of order \( \rho \) and lower order \( \lambda \). S. M. Shah [2] has shown that

\[
\liminf_{r \to \infty} [\mu(r)]^{1/r} \leq e^{1/\rho},
\]

(1)

\[
\limsup_{r \to \infty} [\mu(r)]^{1/r} \geq e^{1/\lambda},
\]

where \( \mu(r) \) denotes the maximum term of \( \sum |a_n| r^n \) and \( v(r) \) is the largest integer \( p \) for which \( \mu(r) = |a_p| r^p \).

The object of the present note is to obtain a sharper form of (1) for those entire series which possess Hadamard gaps. For this purpose let the subsequence \( \{a_{n_m}\} \) contain all the nonvanishing terms of \( \{a_n\} \), and suppose that

\[
\liminf_{m \to \infty} \frac{p_{m+1}}{p_m} \geq 1 + \theta > 1.
\]

(2)

We shall prove the following

**Theorem.** Suppose \( \sum a_{n_m} z^{n_m} \) is an entire series of order \( \rho \) and lower order \( \lambda \) whose gaps satisfy (2). Then

\[
\liminf_{r \to \infty} [\mu(r)]^{1/r} \leq \alpha^{1/\rho},
\]

(3)

\[
\limsup_{r \to \infty} [\mu(r)]^{1/r} \geq \beta^{1/\lambda},
\]

where

\[ \alpha = (1 + \theta)^{1/\theta} \]

and

\[ \beta = (1 + \theta)^{(1+\theta)/\theta}. \]

We call attention to the fact that a series which satisfies (2) need not be of irregular growth; much larger gaps are needed [3] to insure that \( \lambda < \rho \).

**Proof.** The function \( v(r) \) is a nondecreasing step function which is continuous from the right and assumes only nonnegative integer

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values. Therefore there is a nondecreasing sequence \( \{ R_k \} \) which \( \nu(r) \) "counts," i.e.,

\[
\nu(r) = \sum_{R_k \leq r} 1.
\]

For convenience we assume that \( R_1 = 1 \). No generality is lost since this is equivalent to requiring that

\[
|a_0| = \max_{\rho \geq 1} |a_\rho|.
\]

For each \( k \geq 1 \), let

\[
\log i + \cdots + \log \rho \quad (4a)
\]

and

\[
\log k + \cdots + \log \rho \quad (4b)
\]

In addition to satisfying (2), we assume the sequence

\[
p_0, p_1, \ldots, p_m, \ldots
\]

is such that \( p_0 = 0 \) and \( p_1 = 1 \). For notational convenience we shall always denote \( p_m \) by \( n \).

The following relations are easily verified by examining the local minima and maxima of the quantities involved:

\[
\liminf_{m \to \infty} \frac{\log \mu(r)}{\nu(r)} = \liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} = \frac{1}{\rho},
\]

\[
\limsup_{m \to \infty} \frac{\log R_n}{\log \nu(r)} = \limsup_{r \to \infty} \frac{\log r}{\log \nu(r)} = \frac{1}{\lambda}.
\]

We shall also need estimates for the quantities

\[
A(n) = 1 + \sum_{j=1}^{m-1} \left[ \frac{\rho^{j+1}}{\rho^j} - 1 \right]
\]

and
For this purpose let

\[ x_j = \frac{p_{j+1}}{p_j} - 1, \quad j = 1, 2, 3, \ldots \]

Then

\[ \log n = \sum_{j=1}^{m-1} \log(1 + x_j). \]

From (2) and the fact that \((1/x)\log(1+x)\) is a decreasing function, we obtain

\[ \limsup_{m \to \infty} \frac{\log n}{A(n)} \leq \frac{\log(1 + \theta)}{\theta}. \]

A similar argument shows that

\[ \liminf_{m \to \infty} \frac{\log n}{B(n)} \geq \frac{(1 + \theta) \log(1 + \theta)}{\theta}. \]

Having taken care of the above preliminaries, we turn now to the main body of the proof. Inverting the systems of equations (4a) and (4b) yields (since \(R_1 = 1\))

\[ \log R_n = t_n + \sum_{k=2}^{n} \frac{t_k}{k - 1} \quad (5a) \]

and

\[ \log R_{n+1} = u_n + \sum_{k=1}^{n-1} \frac{u_k}{k + 1}. \quad (5b) \]

The values assumed by \(v(r)\) are all terms of \(\{p_m\}\); therefore

\[ \log R_k = \log R_{p_j+1}, \quad p_j < k < p_{j+1}, \]

from which it follows that

\[ t_k = \frac{p_{j+1}}{k} t_{p_{j+1}}, \quad p_j < k \leq p_{j+1}, \]

and

\[ u_k = \frac{p_j}{k} u_{p_j}, \quad p_j \leq k < p_{j+1}. \]
Substituting these expressions in (5a) and (5b), we obtain

\[
(6a) \quad \log R_n = t_n + \sum_{j=1}^{m-1} \frac{p_{j+1}}{p_j} \left[ \frac{1}{p_j} - 1 \right]
\]

and

\[
(6b) \quad \log R_{n+1} = u_n + \sum_{j=1}^{m-1} u_{p_j} \left[ 1 - \frac{p_j}{p_{j+1}} \right].
\]

From (6a) and (6b) it follows (cf. [1, p. 52, Theorem 9]) that

\[
(7a) \quad \liminf_{m \to \infty} t_n \leq \liminf_{m \to \infty} \frac{\log R_n}{A(n)}
\]

and

\[
(7b) \quad \limsup_{m \to \infty} u_n \geq \limsup_{m \to \infty} \frac{\log R_{n+1}}{B(n)}.
\]

From (7a) we have

\[
\liminf_{r \to \infty} \frac{\log \mu(r)}{v(r)} \leq \liminf_{m \to \infty} \frac{\log R_n}{A(n)} \leq \left[ \liminf_{m \to \infty} \frac{\log R_n}{\log n} \right] \left[ \limsup_{m \to \infty} \frac{\log n}{A(n)} \right] \leq \frac{\log(1 + \theta)}{\rho \theta}.
\]

Therefore

\[
\liminf_{r \to \infty} \left[ \mu(r) \right]^{1/v(r)} \leq \alpha^{1/\rho}.
\]

The remaining portion of the theorem follows similarly from (7b).

We note that \( \alpha \) and \( \beta \) tend respectively to 1 and \( \infty \) as \( \theta \to \infty \). In conjunction with our theorem this remark implies the following

**Corollary.** Suppose that \( \sum a_m e^{p_m} \) is an entire function of positive finite order, and

\[
\lim_{m \to \infty} \frac{p_{m+1}}{p_m} = \infty.
\]

Then

\[
\liminf_{r \to \infty} \left[ \mu(r) \right]^{1/v(r)} = 1
\]
and
\[ \limsup_{r \to \infty} [\mu(r)]^{1/r} = \infty. \]

References


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MIXED BOUNDARY-VALUE PROBLEMS IN THE PLANE

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Let \( R \) be a region in the plane bounded by a simple analytic curve \( C \) composed of \( N \) arcs \( C_1 \cdots C_N \). Let \( a_m, b_m, f_m \) be analytic functions on \( C_m \). Suppose \( q(x, y) \) is non-negative in \( R \). The mixed boundary-value problems discussed here require the determination of a solution of
\[
\begin{align*}
\text{(E)} & \quad \Delta u - qu = 0 \quad \text{in } R, \\
\text{(A)} & \quad a_m u_n - b_m u = f_m \quad \text{on } C_m,
\end{align*}
\]
\( n \) the exterior normal. The problem is called regular if on each \( C_m \) either
\[
\begin{align*}
\text{(i)} & \quad a_m > 0, \quad b_m \geq 0 \\
\text{(ii)} & \quad a_m \equiv 0, \quad b_m > 0.
\end{align*}
\]

This note presents an existence theorem based on integral equations. The method is an extension of the solution of the Dirichlet problem by simple layers as in [1] and [4]. It is intended also to provide information as to the behavior of \( u \) at the ends of the \( C_m \).

**Theorem 1.** Every regular mixed problem has a unique solution.

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