CHANGES OF TOPOLOGY AND FIXED POINTS FOR MULTI-VALUED FUNCTIONS

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1. **Introduction.** Let (X, τ) be a topological space and let $S(X)$ be the set of nonempty closed subsets of X. Endow $S(X)$ with the Vietoris finite topology [1]. A basic open set in this topology is a set of the form

$$\langle U_1, \cdots, U_n\rangle = \left\{A \in S(X): A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for } 1 \leqq i \leqq n\right\}$$

where $U_1, \cdots, U_n$ are members of τ. Let $F: X \rightarrow Y$ be a set-valued function on X into Y such that $F(x)$ is closed for each $x \in X$. Then F induces a single-valued function $f: X \rightarrow S(Y)$ on X into $S(Y)$ by $f(x) = F(x)$. Then F is continuous if and only if f is continuous with respect to the finite topology for $S(Y)$. Let $\mathcal{S}$ be a collection of subsets of X. The set $\mathcal{S}$ generates a topology for X by: If $A \subset X$, a point $x \in X$ is an $\mathcal{S}$-limit point for A in case every open set containing x contains a member of $\mathcal{S}$ which meets both $\{x\}$ and $A - \{x\}$; see Young [2]. The topology generated by $\mathcal{S}$ is called the $\mathcal{S}$-topology. We shall obtain a generalization of a theorem of Young [2] on the continuity of functions under changes of topology to include multi-valued functions with each set $F(x)$ finite. This result is then applied to derive a fixed-point theorem for such functions.

In this paper we shall assume that all spaces are Hausdorff.

2. **Finite-valued functions.** A set-valued function $F: X \rightarrow Y$ such that for each $x \in X$ the set $F(x)$ is finite is said to be *finite-valued* or to have finite images. We let $N(F(x))$ denote the number of elements in the set $F(x)$.

In the following, $\mathcal{S}$ will designate a collection of subsets of X and $\mathcal{S}'$ will designate a collection of subsets of X or Y, whichever is appropriate.

First we state a lemma on the continuity of finite-valued functions. The proof of this lemma follows directly from the definition of continuity and from the fact that we have assumed each space to be Hausdorff.

Presented to the Society, August 29, 1963 under the title of *Changes of topology and fixed points for a class of multi-valued functions*; received by the editors November 22, 1963.

LEMMA 1. *Let* $F: X \to Y$ *be a finite-valued function. Let* x *be any element of* X, *with* $F(x) = \{y_1, \cdots, y_k\}$. *Let* $V_1, \cdots, V_k$ *be any collection of disjoint open sets such that* $y_i \in V_i$ *for* $i = 1, \cdots, k$. *Then* F *is continuous at* x *if and only if there exists an open set* U *containing* x *such that for all* $x' \in U$, $F(x') \subset \bigcup_{i=1}^k V_i$ *and* $F(x') \cap V_i \neq \emptyset$, $i = 1, \cdots, k$.

We shall use the following condition.

CONDITION I. Let $\mathcal{S}$, $\mathcal{S}'$ be collections of subsets of X and Y, respectively. Let $F: X \to Y$ be any continuous finite-valued function. If, for each $S \in \mathcal{S}$, there exists an n and subsets $A_1, \cdots, A_n$ of Y such that (i) $F(S) = A_1 \cup \cdots \cup A_n$, (ii) $F(x) \cap A_i \neq \emptyset$, $x \in S$ and $1 \leqq i \leqq n$, (iii) $y_1, y_2 \in A_j$, $1 \leqq j \leqq n$, implies there is an $S' \in \mathcal{S}'$ such that $y_1, y_2 \in S' \subset A_j$, then $\mathcal{S}$ and $\mathcal{S}'$ are said to satisfy Condition I.

THEOREM 2. *Let* $F: X \to Y$ *be finite-valued and continuous with respect to the original topologies. If* $\mathcal{S}$ *and* $\mathcal{S}'$ *satisfy Condition* I *and if* $\mathcal{S}$ *is a base for the* $\mathcal{S}$*-topology on* X, *then* F *is continuous with respect to the* $\mathcal{S}$*- and* $\mathcal{S}'$*-topologies.*

PROOF. Let $x_0 \in X$ and let $F(x_0) = \{y_1, \cdots, y_k\}$. Let $V_1, \cdots, V_k$ be disjoint $\mathcal{S}'$-open subsets of $y_1, \cdots, y_k$. Let $V_1', \cdots, V_k'$ be subsets of $y_1, \cdots, y_k$ which are open in the original topology for Y with $V_i \subset V_i'$. Since V_i' is also $\mathcal{S}'$-open, we may assume that $V_i' \cap V_j' = \emptyset$ if $i \neq j$.

Further, since $\bigcup_{i=1}^k V_i \subset \bigcup_{i=1}^k V_i'$ and $\bigcup_{i=1}^k V_i$ is $\mathcal{S}'$-open we may assume that if $S' \in \mathcal{S}'$, $F(x_0) \cap S' \neq \emptyset$, and $S' \subset \bigcup V_i'$, then $S' \subset \bigcup_{i=1}^k V_i$. Since F is continuous with respect to the original topologies, there exists an open set U containing x_0 such that $F(U) \subset \bigcup_{i=1}^k V_i'$ and such that $x' \in U$ implies $F(x') \cap V_i' \neq \emptyset$, $1 \leqq i \leqq k$. Let $x_0 \in S \subset U$ (such an S exists since $\mathcal{S}$ is a basis for the $\mathcal{S}$-topology and U is $\mathcal{S}$-open). Let $F(S) = A_1 \cup \cdots \cup A_n$ from Condition I. Then, if $y_1, y' \in A_j$, there is an S' such that $y_1, y' \in S' \subset A_j$ but $A_j \subset \bigcup_{i=1}^k V_i'$. Therefore $A_j \subset \bigcup_{i=1}^k V_i$. If $A_j = \{y\}$, then $y \in F(x_0) \subset \bigcup_{i=1}^k V_i$. Thus $F(S) \subset \bigcup_{i=1}^k V_i$. Finally, if $x' \in S$, $F(x') \cap V_i \neq \emptyset$ since $F(x') \cap V_i' \neq \emptyset$ and V_i' meets only V_i. Thus by Lemma 1, F is $\mathcal{S}$- and $\mathcal{S}'$-continuous.

We next state necessary conditions for $\mathcal{S}$ to be a base for the $\mathcal{S}$-topology.

DEFINITION 1. A collection $\mathcal{S}$ of subsets of X is called a *basis set* if and only if (i) each open subset of X is the union of members of $\mathcal{S}$; (ii) $\mathcal{S}$ is closed under finite intersections; (iii) for each $S \in \mathcal{S}$ there is an open set U such that S is a maximal member of $\mathcal{S}$ contained in U.

LEMMA 3. *If* $\mathcal{S}$ *is a basis set, then* $\mathcal{S}$ *is a base for the* $\mathcal{S}$*-topology.*

PROOF. First we show that each S in $\mathcal{S}$ is $\mathcal{S}$-open. Let $x \in X$ and let V be the open set of (iii). Let U be an open set containing x; we may assume that $U \subset V$. Let $x \in S' \subset U$. Since $S' \subset V$ and $S' \cap S \neq \emptyset$, $S' \subset S$ and hence x is not a limit point of $X - S$. Thus S is open. Let V be an $\mathcal{S}$-open set and let $x \in V$. Since x is not an $\mathcal{S}$-limit point of $X - V$, there exists an open set U containing x such that any member of $\mathcal{S}$ which contains x and which is contained in U is contained in V. By (i) at least one such S exists. Thus with (ii), $\mathcal{S}$ is a base for the $\mathcal{S}$-topology.

We also need the following lemma from [3].

LEMMA 4. *Let $F: X \to Y$ be continuous and finite-valued. If $K \subset X$ is connected, then $F(K)$ has at most n components, where*

$$n = \min \{N(F(x)) : x \in K\}.$$

If C is a component of $F(K)$, then $F(x) \cap C \neq \emptyset$ for all $x \in K$.

Following [4], a *topological chain* or, simply, *chain* is a compact connected set which has exactly two non-cutpoints. These two points are the endpoints of the chain. We denote a chain with endpoints a, b by $[a, b]$. We will agree to consider a set of just one point a as a chain, with notation $[a, a]$. A space in which any two distinct points are the endpoints of at most one chain is called *acyclic.* A set in which any two points are the endpoints of at least one chain is said to be *topologically chained* or *chained.*

DEFINITION 2. A space X is said to have *chained components* if and only if for each $x \in X$ and each open set U containing x there is an open set V such that $x \in V \subset U$ and the component of V containing x is chained, or is $\{x\}$.

LEMMA 5. *Let $F: X \to Y$ be continuous and finite-valued. Let Y have chained components. If C is a topological chain in X, then the components of $F(C)$ are topologically chained subsets of Y.*

PROOF. Let $C = [a, b]$. Let $\leqq$ be the cutpoint order for C with a the least element. We shall show that each member of $F(C)$ is chained to some member of $F(a)$. Let $C_1 = \{x \in C$: each $y \in F(x)$ is chained to a member of $F(a)\}$. By convention, $[x, x]$ is a chain, so $a \in C_1$. Let $x_0 = \sup C_1$. To see that $x_0 \in C_1$, let $y \in F(x_0)$ and let V be an open set containing y such that the component $K(y)$ of V which contains y is chained and such that $(F(x_0) - \{y\}) \cap \overline{V} = \emptyset$. Since F is continuous, there is an $x' \in C$ such that $a \leqq x' < x_0$ and for $x' < x < x_0$, $F(x) \cap V \neq \emptyset$. Thus, by Lemma 4, $F(x) \cap K(y) \neq \emptyset$ and y is chained to a member of $F(a)$. On the other hand, if $x_0 < b$, there exists an x' such that $x_0 < x' \leqq b$ and such that $x_0 < x < x'$ implies that $F(x) \cap K(y) \neq \emptyset$. Then $x' \in C_1$, contrary to the definition of x_0. Further, a slight modification of the above argument implies that if $y \in F(a)$ and $x \in [a, b]$, then there exists $y' \in F(x)$ such that y' is chained to y.

Similarly, if Y is a metric space we have as a result of a lemma in [**3**] that

LEMMA 6. *Let Y be a metric space and let C be an arc in X. If $F: X \rightarrow Y$ is a continuous, finite-valued function, then the components of $F(C)$ are arcwise connected.*

A chain component of a set is a subset which is maximal with respect to being chained. Let X be an acyclic space and let $\mathcal{S}$ be the collection of chain components of open sets and let $\mathcal{S}'$ be the collection of chains of X together with the singletons. The topology generated by $\mathcal{S}$ is called the chain topology.

LEMMA 7. *The topologies generated by $\mathcal{S}$ and $\mathcal{S}'$ are equivalent and $\mathcal{S}$ is a base for the chain topology.*

PROOF. That $\mathcal{S}$ and $\mathcal{S}'$ generate equivalent topologies is a direct calculation. Further, $\mathcal{S}$ is a basis set and thus, by Lemma 3, $\mathcal{S}$ is a base for the chain topology.

If X is an acyclic space, and if $\mathcal{S}$ and $\mathcal{S}'$ are as in Lemma 7, then Lemmas 5 or 6 together with Lemma 7 and Theorem 2 give:

THEOREM 8. *Let (X, τ) be an acyclic space with chained components. Let $\mathcal{S}$ be the collection of chained components of open sets. If $F: X \rightarrow X$ is τ-continuous, and finite-valued, then F is $\mathcal{S}$-continuous.*

THEOREM 9. *Let (X, τ) be an acyclic metric space. Let $\mathcal{S}$ be the collection of arcwise connected components of open sets. If $F: X \rightarrow X$ is τ-continuous and finite-valued, then F is $\mathcal{S}$-continuous.*

3. **Fixed-point theorems.** In this section we shall use Theorems 8 and 9 to obtain two generalizations of a fixed-point theorem of Young [**2**].

First we need some results from [**4**]. An acyclic, chained space has an inherent partial order which is obtained by selecting an arbitrary point e as the minimal element and then $x \leqq y$ if and only if $x \in [e, y]$, where $[e, y]$ is the unique chain from e to y. This is the so-called cut-point order. Let $L(x) = \{y \in X: y \leqq x\}$ and let $M(x) = \{y \in X: x \leqq y\}$. A dendritic space is a connected, locally connected space in which each two points can be separated by the omission of a third point. Lemmas 10 and 11 below are from [**4**].

LEMMA 10. *A necessary and sufficient condition that a locally connected space be dendritic is that it admit a partial order satisfying*

(i) *$L(x)$ and $M(x)$ are closed sets for each point x,*

(ii) *if $x<y$, then there exists z such that $x<z$ and $z<y$,*

(iii) *for each x and y, the set $L(x)\cap L(y)$ is nonempty, compact and simply ordered,*

(iv) *for each x, the set $M(x)-x$ is open.*

Also we have:

LEMMA 11. *A necessary and sufficient condition that a Hausdorff space X be acyclic and chained is that it be dendritic in its chain topology.*

Note that if X is a space in which every nest of arcs (or chains) is contained in an arc (or chain), then X is necessarily acyclic.

THEOREM 12a. *The class of arcwise connected, metric spaces in which every nest of arcs is contained in an arc has the fixed-point property for finite-valued functions.*

THEOREM 12b. *The class of topological spaces which are topologically chained, have small chained components, and in which every nest of chains is contained in a chain, has the fixed-point property for finite-valued functions.*

We shall prove 12a. A proof of 12b is then obtained by replacing arcs by chains and using Theorem 8 rather than Theorem 9. We shall also need the following lemma which can be proved in a straightforward way.

LEMMA 13. *Let X be an acyclic chained space with minimal element e. Let $\{x_\alpha\}$ be a net in $[e, x_0]$ which converges to x_0. Let $\{y_\alpha\}$ be a net such that $y_\alpha \geqq x_\alpha$ for each α. If $\{y_\alpha\}$ converges to y_0 in the chain topology, then $y_0 \geqq x_0$.*

PROOF OF 12a. Let e be the minimal element of X in the cutpoint order. Let $F: X\to X$ be a continuous finite-valued function. By Theorem 9, F is also continuous when X has the chain topology. Define a family $\mathcal{S}$ by:

(i) $S\in\mathcal{S}$ if and only if $S=[e, x']$ for some $x'\in X$, and,

(ii) $x\in[e, x']$ implies there is a $y\in F(x)$ such that $y\geqq x$.

The family $\mathcal{S}$ is partially ordered by inclusion, and $S=[e,e]=\{e\}\in\mathcal{S}$. Let $\mathcal{S}_0$ be a linearly ordered subfamily of $\mathcal{S}$. Then by hypothesis there exists an arc $[e, x]$ such that $\bigcup\mathcal{S}_0\subset[e, x]$. Let x_0 be the l.u.b. of $\bigcup\mathcal{S}_0$ in $[e, x]$. We shall show that $[e, x_0]\in\mathcal{S}$. For this it suffices to show that there is a $y\in F(x_0)$ such that $x_0\leqq y$. Let $F(x_0)=\{y_1, \cdots, y_k\}$. Let $V_1, \cdots, V_k$ be disjoint basic chain-open subsets of $y_1, \cdots, y_k$, respectively. We may assume that $x_0 \notin F(x_0)$ and hence that $x_0 \notin \bigcup_{i=1}^k V_i$. Let $\{x_\alpha\}$ be a net in $[e, x_0]$ such that $\{x_\alpha\}$ converges to x_0, and $x_\alpha < x_0$ for each α. Then for each α there is a $y_\alpha \in F(x_\alpha)$ such that $x_\alpha \leqq y_\alpha$. We now assert that there is an i such that the net $\{y_\alpha\}$ is frequently in every chain-open neighborhood (c-neighborhood) of y_i. Suppose $i=1$. Then there exists a subnet $\{y_{\alpha'}\}$ of $\{y_\alpha\}$ which is eventually in every c-neighborhood of y_1 [**5**, p. 70]. Then, by Lemma 13, $x_0 \leqq y_1$. Consequently, each nest in $\mathcal{S}$ has an upper bound and by Zorn's lemma there is a maximal element in $\mathcal{S}$. Let $[e, x_0]$ be a maximal element of $\mathcal{S}$. Let $y_0 \in F(x_0) \cap M(x_0)$. Suppose that $x_0 \notin F(x_0)$. Let (x_0, y_0) be the open arc from x_0 to y_0. Let $x' \in (x_0, y_0)$. Then $M(x') - x'$ is a chain open set containing y_0. Hence, by continuity of F, there exists an x_1, $x_0 < x_1 < x'$, such that, for any $x \in [x_0, x_1]$, $F(x) \cap (M(x') - x') \neq \emptyset$. This contradicts the maximality of $[e, x_0]$. Hence $x_0 \in F(x_0)$.

4. **Functions with infinite images.** In this section we briefly investigate the situation when $F(x)$ may be an infinite set. First we give an example which shows that the conclusion of Theorems 8 and 9 fails in this case.

EXAMPLE 1. Let A be the graph of the curve of $\sin \pi/x$, $0 < x \leqq 1$. Let B be the segment of the y-axis between $y = -2$ and $y = +1$. Let C be the horizontal line segment from $(0, -2)$ to $(1, -2)$, and let D be the vertical segment from $(1, -2)$ to $(1, 0)$. Finally let $X = A \cup B \cup C \cup D$. To define the function we wish, we shall need additional notation.

Let $B_1 = \{(0, y) : -1 \leqq y \leqq 1\}$; let $D_0 = D - \{(1, 0)\}$, and denote a member of X by $x = (a, b)$.

Let $A(a)$ be the portion of A to the left of the line $x = a$. Then, for $x \in D_0$, define $F(x) = A(|b|/2) \cup B_1$, and define $F(1, 0) = B_1$. For $x \in A$ and $1/2 \leqq a \leqq 1$, set

$$F(x) = \{(0, y) : -1 - 2(1 - a) \leqq y \leqq 1 - 4(1 - a)\}$$

and

$$F(x) = \{(0, y) : -2 \leqq y \leqq -1 - 2(1/2 - a)\}$$

if $0 < a \leqq 1/2$. For $x \in B_1$ set $F(x) = \{(0, -2)\}$ and for $x \in B - B_1$ set $F(x) = \{(|1 + b|, -2)\}$. For $x \in C$ and $0 \leqq a \leqq 1/2$ set $F(x) = \{(1, -2 + 4a)\}$. Let $A(a)^{\sim}$ be the intersection of A with the vertical strip determined by the lines $x = 1$ and $x = a$. Then for $x \in C$ and $1/2 \leqq x < 1$ set $F(x) = A(2(1 - a))^{\sim}$.

Then F is continuous and has compact connected images. (In fact we could have, with more complication in notation, had $F(x)$ an arc for each x.) But F is not continuous in the chain topology since any sequence in D which converges to $(1, 0)$ has images that do not converge to the image of $F(1, 0)$ in the chain topology.

With one change the condition given by Connell [6] can be shown to be a sufficient condition for the preservation of continuity of arbitrary multi-valued functions.

THEOREM 14. *Let τ_1 and τ_2 be two Hausdorff topologies for a set X. Suppose that τ_2 is normal, that $\tau_2 \subset \tau_1$, and that the closure of any τ_1-open set is the same in each topology. Then, any function which is continuous with respect to τ_1 is continuous with respect to τ_2.*

PROOF. Let $F: X \to X$ be a τ_1-continuous function on X into X. Let $f: X \to S(X)$ be the map induced by F. Let T_1 and T_2 be the finite topologies for $S(X)$ generated by τ_1 and τ_2, respectively. Let $x \in X$ and let $\mathcal{V}$ be a T_2-open subset containing $f(x)$. Since τ_2 is a normal topology, the topology T_2 is regular [1, Theorem 4.9.5]. Thus let $\mathcal{V}_1 \subset \mathcal{V}_1 \subset \mathcal{V}$ in T_2. Since $\tau_2 \subset \tau_1$, $T_2 \subset T_1$, thus $\mathcal{V}_1$ is τ_1-open and hence $f^{-1}(\mathcal{V}_1)$ is τ_1-open. Set $U = f^{-1}(\mathcal{V}_1)$. Then the closure of U is the same in τ_1 and τ_2, denote this by U^-, and let $U^{-\sim}$ denote the complement of the closure and so forth. Then $x \in U^{-\sim-\sim} \subset U^-$ and $U^{-\sim-\sim}$ is open in τ_1 and in τ_2. Furthermore, the T_1-closure $(\mathcal{V}_1^{-T_1})$ of $\mathcal{V}_1$ is contained in the T_2-closure $(\mathcal{V}_1^{-T_2})$ of $\mathcal{V}_1$. Thus $f^{-1}(\mathcal{V}_1)^{-\tau_2} = f^{-1}(\mathcal{V}_1)^{-\tau_1} \subset f^{-1}(\mathcal{V}^{-T_1}) \subset f^{-1}(\mathcal{V}_1^{-T_2})$.

Hence $f^{-1}(\mathcal{V}_1)^- \subset f^{-1}(\mathcal{V}_1^{-T_2})$. Then

$$f(U^{-\sim-\sim}) \subset f(\overline{U}) = f(f^{-1}({}_1)\mathcal{V}^-) \subset \mathcal{V}_1^{-T_2} \subset \mathcal{V}.$$

Thus f is T_2-continuous and hence F is τ_2-continuous.

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