EQUAL-DIFFERENCE BIB DESIGNS

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A balanced incomplete block design (BIB design) is an arrangement of v elements into b subsets, or blocks, each containing k elements, so that each element appears in r blocks and each pair of elements appears in λ blocks. The numbers (v, k, b, r, λ) are called the parameters of the design. These designs, particularly those with $r \leqq 10$, are of interest to statisticians in connection with the design of experiments, and, accordingly, tables have been prepared in which such designs are listed [1], [2], [3].

The purpose of this paper is to exhibit a doubly-infinite family of designs, whose law of formation is quite simple, but which fail to appear in any of the above-mentioned tables. The parameters of these designs are

$$\begin{aligned} v &= v, \\ k &= k, \\ b &= v(v-1)/2, \\ r &= k(v-1)/2, \\ \lambda &= k(k-1)/2, \end{aligned} \tag{1}$$

where, if v is a prime, k can be any number less than v, while if v is composite, k can be any number not greater than the smallest prime factor of v.

The elements of a design of this family are the residue classes modulo v. There are b/v initial blocks, such that v blocks of the design can be obtained from each initial block by adding to every element of it, in turn, each of the residue classes modulo v. The difference between any two consecutive elements of an initial block is a constant for that initial block. It is for this reason that the designs are called equal-difference designs. If $k=2$, the equal-difference design is simply the unreduced design obtained by taking all possible pairs of v elements. If $k>2$, which is possible only for v odd, then the design will not be unreduced.

The initial blocks of this design are

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$$(0\ 1\ \cdots\ k-1)\ (0\ 2\ \cdots\ 2(k-1))\ \cdots \tag{2}$$
$$\left(0\ \frac{v-1}{2}\ \cdots\ \frac{(k-1)(v-1)}{2}\right) \bmod v \text{ for odd } v.$$

Although this representation of the initial blocks is not applicable to even values of v, the only equal-difference designs for which v is even are the unreduced designs with $k=2$, and these are trivially included in the set of equal-difference designs, since an ordered set consisting of only two numbers is necessarily an arithmetic progression. Accordingly, let us confine our attention to odd values of v, for which (2) is valid.

THEOREM. *If v is odd and $k \leqq p$, where p is the smallest prime factor of v, the result of adding, in turn, to each of the initial blocks of* (2), *each of the residue classes modulo v, is a balanced incomplete block design with parameters* (1).

PROOF. It is necessary to establish both that the elements of each initial block are distinct and that every pair of elements occurs in precisely λ blocks, in order to prove this theorem. First, in order prove the distinctness of the elements of each initial block, assume the contrary, namely, that for some pair of numbers (c, d), where $0<c<k$ and $d \leqq (v-1)/2$, the $(c+1)$st element of the dth initial block is 0. But this would mean that $cd=0$ modulo v, and therefore c is a factor of v. But this is impossible since $c \leqq k \leqq p$, where p is the smallest prime factor of v.

To determine the number of blocks in which a pair of elements appears, let d be the difference between these two elements, and multiply the initial blocks by d modulo v. This produces a permutation of the sets of blocks generated by the initial blocks, and multiplies the difference between any two elements by d. It follows that the number of blocks containing two elements differing by 1 is equal to the number of blocks containing two elements differing by d. Since this is true no matter what value of d is chosen, the number of blocks containing any two elements is a constant λ. The value of λ will be equal to the number of pairs of elements in an initial block, multiplied by the number of initial blocks, and divided by the number of distinct values that d can have, where two values of d are considered distinct if neither their sum nor their difference is zero modulo v. But since the number of distinct values that d can have is equal to the number of initial blocks, the value of λ is equal to the number of pairs of elements in a single initial block, which is $k(k-1)/2$. This completes the proof.

EXAMPLE 1. $v=7$, $k=3$, $b=21$, $r=9$, $\lambda=3$.

INITIAL BLOCKS. (0 1 2) (0 2 4) (0 3 6) mod 7.

This design contains all possible sets of 3 elements except those generated by the two initial blocks (0 1 3) and (0 2 3), each of which generates a design with parameters $v=b=7$, $k=r=3$, $\lambda=1$, corresponding to the well-known geometry on seven points.

EXAMPLE 2. $v=11$, $k=4$, $b=55$, $r=20$, $\lambda=6$.

INITIAL BLOCKS. (0 1 2 3) (0 2 4 6) (0 3 6 9) (0 4 8 1) (0 5 10 4) mod 11.

This is a solution to the last set of parameters listed in [4], and it was, in fact, the attempt to construct this design that led to the discovery of the equal-difference designs.

As the referee pointed out, [5] contains necessary and sufficient conditions for the existence of BIB designs for $k=3$ and for $k=4$. In particular, a BIB design having the same parameters as Example 2 above can be found in [5, §(6.7.4), p. 373].

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