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DIFFERENTIABLE ACTIONS OF COMPACT ABELIAN LIE GROUPS ON S^n

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1. **Introduction.** In [9] P. A. Smith raises the following question: If a finite group G acts effectively on the n-sphere S^n, must there also be some effective orthogonal action of G on S^n? Stated another way, must all finite groups acting effectively on S^n be isomorphic to subgroups of the orthogonal group $O(n+1)$? Smith has answered this question in the affirmative for the case where G is an elementary p-group [8], [9]. The Corollary to Theorem 2 of this paper settles the case where G is a compact abelian Lie group (in particular, a finite abelian group) and the action is assumed differentiable.

The proof of our main result is immediate if one assumes the existence of a fixed point, as evidenced by the following result which utilizes Bochner's theorem on local linearity about a fixed point.

Theorem 1. *Let G be a compact Lie group operating effectively and differentiably on a differentiable n-manifold X. If there exits a point x_0 left fixed by every element of G, then G is isomorphic to a subgroup of $O(n)$.*

Proof. By Bochner's theorem [5, p. 206], we may assume G acts orthogonally (but not necessarily effectively) on some small closed n-cell D with center x_0. G leaves bdy $D=S^{n-1}$ invariant. If G is not effective on S^{n-1}, then there must be a homeomorphism g_0 of finite order in G which leaves S^{n-1} pointwise fixed. Since g_0 acts linearly on D and leaves x_0 fixed, it must also leave D pointwise fixed. By Newman's theorem [5, p. 223], g_0 must leave X pointwise fixed, violating the effectiveness of G on X. Hence G acts orthogonally and effectively on S^{n-1}, and the theorem follows.

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2. **Definitions and preliminaries.** An action of a transformation group G on a space X is said to be *effective* if $gx = x$ for all x in X implies $g = e$, the identity element of G. All spaces considered will be compact Hausdorff spaces and the usual Čech cohomology will be used. Let Z_p denote the additive group of integers modulo a prime p. Our definition of a *cohomology n-manifold over* Z_p and of *cohomology dimension over* Z_p will be that given in [1]. An *elementary p-group of rank k* is a group isomorphic to the direct sum of k copies of Z_p. Smith [7] has shown that if an elementary p-group G acts effectively on a connected cohomology n-manifold X over Z_p, then each component of $F(G, X)$, the fixed-point set of G on X, is a connected cohomology r-manifold over Z_p with $r < n$ for each r. In fact, if p is an odd prime, each $n - r$ must be even. A *generalized cohomology n-sphere over* Z_p is a cohomology n-manifold over Z_p which has the global cohomology, coefficient group Z_p, of S^n. Results of Smith [6], [7] show that if an elementary p-group G acts effectively on a generalized cohomology n-sphere X over Z_p, then $F(G, X)$ is a generalized cohomology r-sphere over Z_p, $r < n$. We shall need the following result from [3]:

LEMMA 1. *Let G be an elementary p-group of rank k acting effectively on a connected n-manifold X. Suppose $F(G, X)$ is nonempty and of cohomology dimension r. Then*

$$k \leqq \begin{cases} \dfrac{n-r}{2} & \text{for } p \neq 2, \\ n - r & \text{for } p = 2. \end{cases}$$

3. **Main results.**

THEOREM 2. *Let G be isomorphic to the direct sum of k copies of Z_2 and l copies of Z_{2p}, p an odd prime. If G acts effectively and differentiably on S^n, then*

$$k + 2l \leqq n + 1.$$

PROOF. By the above-mentioned Smith result [8], [9], we may assume $k, l \geqq 1$. We shall suppose $k + 2l = n + 2$ and arrive at a contradiction.

There exists a subgroup H of G which is an elementary 2-group of rank $k + l$. We consider the action of H on S^n. As usual, the *isotropy subgroup* of H at a point x in X is defined as the subgroup of H consisting of all elements which leave x fixed. By the results of [8] or [4], there must exist an isotropy subgroup H_1 of H of rank $k + l - 1$. Since H_1 is an isotropy subgroup, $F(H, X)$ must be a proper subset of $F(H_1, X)$. Proceeding inductively, we may construct the following two decompositions

(1) $$H = H_0 \supset H_1 \supset \cdots \supset H_{k+l-1} \supset H_{k+l} = e,$$

(2) $$\emptyset \subseteq F_0 \subset F_1 \subset \cdots \subset F_{k+l-1} \subset F_{k+l} = S^n,$$

where

(i) H_i, $0 \leqq i \leqq k+l$, is a subgroup of H of rank $k+l-i$ which operates on S^n with fixed-point set F_i.

(ii) Each F_i is a generalized cohomology sphere over Z_2.

(iii) Each F_i is a compact manifold. This fact follows from the differentiability of the action by applying Bochner's theorem. Moreover, by (ii), each F_i of positive dimension is connected.

(iv) $\dim F_i < \dim F_{i+1}$, $0 \leqq i \leqq k+l-1$.

We proceed to investigate the decomposition (2) of S^n. From $\emptyset$ to S^n, there are exactly $(n+2)-(k+l+1)=l-1$ gaps in dimension, that is, dimensions not assumed by the F_i's. It follows that $\dim F_{i_0}$ is even (including 0) for some i_0, with $\dim F_{i_0} \leqq 2l-2$; for, otherwise, there would have to be at least l gaps. Select then the first i_0 with $\dim F_{i_0}$ even. As $k \geqq 1$, we have $\dim F_{i_0} \leqq 2l-2 < n$.

There exists a subgroup K of G which is an elementary p-group of rank l. The action of K on S^n leaves F_{i_0} invariant. Since F_{i_0} is a generalized cohomology sphere over Z_2, the Euler characteristic, $\chi(F_{i_0})$, of F_{i_0}, which is independent of the coefficient field, must be equal to 2. Hence, $\chi(F_{i_0})$ is not congruent to 0 modulo p, and by an easy extension of Floyd's result in [2] to actions of elementary p-groups, the action of K on F_{i_0} must have a fixed point. By Lemma 1, there exists a subgroup T of K such that each element of T leaves F_{i_0} pointwise fixed and such that K/T is effective on F_{i_0} with

$$\Gamma(K/T) = \text{rank }(K/T) \leqq \frac{\dim F_{i_0}}{2}\cdot$$

We shall say that T is *completely noneffective* on F_{i_0}. Now

$$\Gamma(T) \geqq \Gamma(K) - \frac{\dim F_{i_0}}{2} = l - \frac{\dim F_{i_0}}{2} \geqq l - \frac{2l-2}{2} = 1.$$

We next construct a decomposition of T,

(3) $$T = T_{i_0} \supseteq T_{i_0+1} \supseteq \cdots \supseteq T_{k+l} \supset e,$$

such that T_j, $i_0 \leqq j \leqq k+l$, is completely noneffective on F_j. As T_{k+l} is a nontrivial subgroup of G which is noneffective on $F_{k+l} = S^n$, we obtain a contradiction to the effectiveness of G.

We proceed with the construction of (3). Now at least $(\dim F_{i_0})/2$ gaps have been used in arriving to F_{i_0} in (2). Therefore there exist at most

$$(l-1)-\frac{\dim F_{i_0}}{2}$$

gaps from F_{i_0} to F_{k+l} in (2). Let $N=\dim F_{i_0+1}-\dim F_{i_0}$ and consider the following two cases.

(i) $N=1$. Now T leaves the compact connected manifold F_{i_0+1} invariant with fixed-point set containing F_{i_0}. Since $N=1$ and each element of T is of prime order p, p odd, we have that T is completely noneffective on F_{i_0+1} due to above-mentioned parity restrictions. In this case, choose $T_{i_0+1}=T$.

(ii) $N\geqq 2$. In this case, there are $N-1$ gaps from F_{i_0} to F_{i_0+1} in (2). Again, T leaves F_{i_0+1} invariant with fixed-point set F containing F_{i_0}. By Lemma 1, there exists a subgroup T_{i_0+1} of T which is completely noneffective on F_{i_0+1} with

$$\begin{aligned}\Gamma(T/T_{i_0+1}) &\leqq \frac{\dim F_{i_0+1}-\dim F}{2}\\ &\leqq \frac{\dim F_{i_0+1}-\dim F_{i_0}}{2}=\frac{N}{2}\,.\end{aligned}$$

Consequently,

$$\Gamma(T_{i_0+1}) \geqq \Gamma(T)-\frac{N}{2} \geqq \Gamma(T)-(N-1).$$

We see that each gap results in reducing the rank of T to that of T_{i_0+1} by at most one.

Proceeding inductively, and recalling that there are at most

$$(l-1)-\frac{\dim F_{i_0}}{2}$$

gaps from F_{i_0} to F_{k+l}, we obtain

$$\begin{aligned}\Gamma(T_{k+l}) &\geqq \Gamma(T)-\left[(l-1)-\frac{\dim F_{i_0}}{2}\right]\\ &\geqq \left[l-\frac{\dim F_{i_0}}{2}\right]-\left[(l-1)-\frac{\dim F_{i_0}}{2}\right]=1.\end{aligned}$$

Corollary. *Let G be a compact abelian Lie group acting effectively* *and differentiably on S^n. Then G is isomorphic to a subgroup of $O(n+1)$.*

PROOF. G is the direct sum of a q-torus T^q and a finite abelian group R. Choose a minimal set of generators of R, let $g(R)$ denote the total number of these generators and $h(R)$ denote the number of these generators of order 2. Suppose, first, that $h(R)=0$. Then there exists for some odd prime p a subgroup G' of G with G' an elementary p-group of rank $q+g(R)$. Then, by Smith [8], [9],

$$q + g(R) \leqq \left[\frac{n+1}{2}\right]$$

and the Corollary follows. Suppose then that $h(R) \geqq 1$. Then there exists a subgroup G'' of G with G'' isomorphic to the direct sum of $h(R)$ copies of Z_2 and $[q+g(R)-h(R)]$ copies of Z_{2p}, for some odd prime p. The Corollary now follows from Theorem 2.

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