

4. ———, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 105–108.

5. ———, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.

INSTITUTE FOR DEFENSE ANALYSES AND  
UNIVERSITY OF VIRGINIA

---

## DIFFERENTIABLE ACTIONS OF COMPACT ABELIAN LIE GROUPS ON $S^n$

L. N. MANN

1. **Introduction.** In [9] P. A. Smith raises the following question: If a finite group  $G$  acts effectively on the  $n$ -sphere  $S^n$ , must there also be some effective orthogonal action of  $G$  on  $S^n$ ? Stated another way, must all finite groups acting effectively on  $S^n$  be isomorphic to subgroups of the orthogonal group  $O(n+1)$ ? Smith has answered this question in the affirmative for the case where  $G$  is an elementary  $p$ -group [8], [9]. The Corollary to Theorem 2 of this paper settles the case where  $G$  is a compact abelian Lie group (in particular, a finite abelian group) and the action is assumed differentiable.

The proof of our main result is immediate if one assumes the existence of a fixed point, as evidenced by the following result which utilizes Bochner's theorem on local linearity about a fixed point.

**THEOREM 1.** *Let  $G$  be a compact Lie group operating effectively and differentiably on a differentiable  $n$ -manifold  $X$ . If there exists a point  $x_0$  left fixed by every element of  $G$ , then  $G$  is isomorphic to a subgroup of  $O(n)$ .*

**PROOF.** By Bochner's theorem [5, p. 206], we may assume  $G$  acts orthogonally (but not necessarily effectively) on some small closed  $n$ -cell  $D$  with center  $x_0$ .  $G$  leaves  $\text{bdy } D = S^{n-1}$  invariant. If  $G$  is not effective on  $S^{n-1}$ , then there must be a homeomorphism  $g_0$  of finite order in  $G$  which leaves  $S^{n-1}$  pointwise fixed. Since  $g_0$  acts linearly on  $D$  and leaves  $x_0$  fixed, it must also leave  $D$  pointwise fixed. By Newman's theorem [5, p. 223],  $g_0$  must leave  $X$  pointwise fixed, violating the effectiveness of  $G$  on  $X$ . Hence  $G$  acts orthogonally and effectively on  $S^{n-1}$ , and the theorem follows.

---

Received by the editors March 10, 1964.

2. **Definitions and preliminaries.** An action of a transformation group  $G$  on a space  $X$  is said to be *effective* if  $gx = x$  for all  $x$  in  $X$  implies  $g = e$ , the identity element of  $G$ . All spaces considered will be compact Hausdorff spaces and the usual Čech cohomology will be used. Let  $Z_p$  denote the additive group of integers modulo a prime  $p$ . Our definition of a *cohomology  $n$ -manifold over  $Z_p$*  and of *cohomology dimension over  $Z_p$*  will be that given in [1]. An *elementary  $p$ -group of rank  $k$*  is a group isomorphic to the direct sum of  $k$  copies of  $Z_p$ . Smith [7] has shown that if an elementary  $p$ -group  $G$  acts effectively on a connected cohomology  $n$ -manifold  $X$  over  $Z_p$ , then each component of  $F(G, X)$ , the fixed-point set of  $G$  on  $X$ , is a connected cohomology  $r$ -manifold over  $Z_p$  with  $r < n$  for each  $r$ . In fact, if  $p$  is an odd prime, each  $n - r$  must be even. A *generalized cohomology  $n$ -sphere over  $Z_p$*  is a cohomology  $n$ -manifold over  $Z_p$  which has the global cohomology, coefficient group  $Z_p$ , of  $S^n$ . Results of Smith [6], [7] show that if an elementary  $p$ -group  $G$  acts effectively on a generalized cohomology  $n$ -sphere  $X$  over  $Z_p$ , then  $F(G, X)$  is a generalized cohomology  $r$ -sphere over  $Z_p$ ,  $r < n$ . We shall need the following result from [3]:

LEMMA 1. *Let  $G$  be an elementary  $p$ -group of rank  $k$  acting effectively on a connected  $n$ -manifold  $X$ . Suppose  $F(G, X)$  is nonempty and of cohomology dimension  $r$ . Then*

$$k \leq \begin{cases} \frac{n-r}{2} & \text{for } p \neq 2, \\ n-r & \text{for } p = 2. \end{cases}$$

### 3. Main results.

THEOREM 2. *Let  $G$  be isomorphic to the direct sum of  $k$  copies of  $Z_2$  and  $l$  copies of  $Z_{2p}$ ,  $p$  an odd prime. If  $G$  acts effectively and differentiably on  $S^n$ , then*

$$k + 2l \leq n + 1.$$

PROOF. By the above-mentioned Smith result [8], [9], we may assume  $k, l \geq 1$ . We shall suppose  $k + 2l = n + 2$  and arrive at a contradiction.

There exists a subgroup  $H$  of  $G$  which is an elementary 2-group of rank  $k + l$ . We consider the action of  $H$  on  $S^n$ . As usual, the *isotropy subgroup* of  $H$  at a point  $x$  in  $X$  is defined as the subgroup of  $H$  consisting of all elements which leave  $x$  fixed. By the results of [8] or [4], there must exist an isotropy subgroup  $H_1$  of  $H$  of rank  $k + l - 1$ . Since  $H_1$  is an isotropy subgroup,  $F(H, X)$  must be a proper subset

of  $F(H_1, X)$ . Proceeding inductively, we may construct the following two decompositions

$$(1) \quad H = H_0 \supset H_1 \supset \cdots \supset H_{k+l-1} \supset H_{k+l} = e,$$

$$(2) \quad \emptyset \subseteq F_0 \subset F_1 \subset \cdots \subset F_{k+l-1} \subset F_{k+l} = S^n,$$

where

(i)  $H_i, 0 \leq i \leq k+l$ , is a subgroup of  $H$  of rank  $k+l-i$  which operates on  $S^n$  with fixed-point set  $F_i$ .

(ii) Each  $F_i$  is a generalized cohomology sphere over  $Z_2$ .

(iii) Each  $F_i$  is a compact manifold. This fact follows from the differentiability of the action by applying Bochner's theorem. Moreover, by (ii), each  $F_i$  of positive dimension is connected.

(iv)  $\dim F_i < \dim F_{i+1}, 0 \leq i \leq k+l-1$ .

We proceed to investigate the decomposition (2) of  $S^n$ . From  $\emptyset$  to  $S^n$ , there are exactly  $(n+2) - (k+l+1) = l-1$  gaps in dimension, that is, dimensions not assumed by the  $F_i$ 's. It follows that  $\dim F_{i_0}$  is even (including 0) for some  $i_0$ , with  $\dim F_{i_0} \leq 2l-2$ ; for, otherwise, there would have to be at least  $l$  gaps. Select then the first  $i_0$  with  $\dim F_{i_0}$  even. As  $k \geq 1$ , we have  $\dim F_{i_0} \leq 2l-2 < n$ .

There exists a subgroup  $K$  of  $G$  which is an elementary  $p$ -group of rank  $l$ . The action of  $K$  on  $S^n$  leaves  $F_{i_0}$  invariant. Since  $F_{i_0}$  is a generalized cohomology sphere over  $Z_2$ , the Euler characteristic,  $\chi(F_{i_0})$ , of  $F_{i_0}$ , which is independent of the coefficient field, must be equal to 2. Hence,  $\chi(F_{i_0})$  is not congruent to 0 modulo  $p$ , and by an easy extension of Floyd's result in [2] to actions of elementary  $p$ -groups, the action of  $K$  on  $F_{i_0}$  must have a fixed point. By Lemma 1, there exists a subgroup  $T$  of  $K$  such that each element of  $T$  leaves  $F_{i_0}$  pointwise fixed and such that  $K/T$  is effective on  $F_{i_0}$  with

$$\Gamma(K/T) = \text{rank } (K/T) \leq \frac{\dim F_{i_0}}{2}.$$

We shall say that  $T$  is *completely noneffective* on  $F_{i_0}$ . Now

$$\Gamma(T) \geq \Gamma(K) - \frac{\dim F_{i_0}}{2} = l - \frac{\dim F_{i_0}}{2} \geq l - \frac{2l-2}{2} = 1.$$

We next construct a decomposition of  $T$ ,

$$(3) \quad T = T_{i_0} \supseteq T_{i_0+1} \supseteq \cdots \supseteq T_{k+l} \supset e,$$

such that  $T_j, i_0 \leq j \leq k+l$ , is completely noneffective on  $F_j$ . As  $T_{k+l}$  is a nontrivial subgroup of  $G$  which is noneffective on  $F_{k+l} = S^n$ , we obtain a contradiction to the effectiveness of  $G$ .

We proceed with the construction of (3). Now at least  $(\dim F_{i_0})/2$  gaps have been used in arriving to  $F_{i_0}$  in (2). Therefore there exist at most

$$(l - 1) - \frac{\dim F_{i_0}}{2}$$

gaps from  $F_{i_0}$  to  $F_{k+l}$  in (2). Let  $N = \dim F_{i_0+1} - \dim F_{i_0}$  and consider the following two cases.

(i)  $N = 1$ . Now  $T$  leaves the compact connected manifold  $F_{i_0+1}$  invariant with fixed-point set containing  $F_{i_0}$ . Since  $N = 1$  and each element of  $T$  is of prime order  $p$ ,  $p$  odd, we have that  $T$  is completely noneffective on  $F_{i_0+1}$  due to above-mentioned parity restrictions. In this case, choose  $T_{i_0+1} = T$ .

(ii)  $N \geq 2$ . In this case, there are  $N - 1$  gaps from  $F_{i_0}$  to  $F_{i_0+1}$  in (2). Again,  $T$  leaves  $F_{i_0+1}$  invariant with fixed-point set  $F$  containing  $F_{i_0}$ . By Lemma 1, there exists a subgroup  $T_{i_0+1}$  of  $T$  which is completely noneffective on  $F_{i_0+1}$  with

$$\begin{aligned} \Gamma(T/T_{i_0+1}) &\leq \frac{\dim F_{i_0+1} - \dim F}{2} \\ &\leq \frac{\dim F_{i_0+1} - \dim F_{i_0}}{2} = \frac{N}{2}. \end{aligned}$$

Consequently,

$$\Gamma(T_{i_0+1}) \geq \Gamma(T) - \frac{N}{2} \geq \Gamma(T) - (N - 1).$$

We see that each gap results in reducing the rank of  $T$  to that of  $T_{i_0+1}$  by at most one.

Proceeding inductively, and recalling that there are at most

$$(l - 1) - \frac{\dim F_{i_0}}{2}$$

gaps from  $F_{i_0}$  to  $F_{k+l}$ , we obtain

$$\begin{aligned} \Gamma(T_{k+l}) &\geq \Gamma(T) - \left[ (l - 1) - \frac{\dim F_{i_0}}{2} \right] \\ &\geq \left[ l - \frac{\dim F_{i_0}}{2} \right] - \left[ (l - 1) - \frac{\dim F_{i_0}}{2} \right] = 1. \end{aligned}$$

**COROLLARY.** *Let  $G$  be a compact abelian Lie group acting effectively*

and differentiably on  $S^n$ . Then  $G$  is isomorphic to a subgroup of  $O(n+1)$ .

PROOF.  $G$  is the direct sum of a  $q$ -torus  $T^q$  and a finite abelian group  $R$ . Choose a minimal set of generators of  $R$ , let  $g(R)$  denote the total number of these generators and  $h(R)$  denote the number of these generators of order 2. Suppose, first, that  $h(R) = 0$ . Then there exists for some odd prime  $p$  a subgroup  $G'$  of  $G$  with  $G'$  an elementary  $p$ -group of rank  $q + g(R)$ . Then, by Smith [8], [9],

$$q + g(R) \leq \left[ \frac{n+1}{2} \right]$$

and the Corollary follows. Suppose then that  $h(R) \geq 1$ . Then there exists a subgroup  $G''$  of  $G$  with  $G''$  isomorphic to the direct sum of  $h(R)$  copies of  $Z_2$  and  $[q + g(R) - h(R)]$  copies of  $Z_{2p}$ , for some odd prime  $p$ . The Corollary now follows from Theorem 2.

#### REFERENCES

1. A. Borel et al., *Seminar on transformation groups*, Princeton Univ. Press, Princeton, N. J., 1960.
2. E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. **72** (1952), 138-147.
3. L. N. Mann and J. C. Su, *Actions of elementary  $p$ -groups on manifolds*, Trans. Amer. Math. Soc. **106** (1963), 115-126.
4. L. N. Mann, *A note on counting isotropy subgroups*, Proc. Amer. Math. Soc. **16** (1965) 476-480.
5. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.
6. P. A. Smith, *Transformations of finite period*, Ann. of Math. (2) **39** (1938), 127-164.
7. ———, *Transformations of finite period*. II, Ann. of Math. (2) **40** (1939), 690-711.
8. ———, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 105-108.
9. ———, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401-415.

INSTITUTE FOR DEFENSE ANALYSES AND  
UNIVERSITY OF VIRGINIA