DISTRIBUTION PROOF OF WIENER'S TAUBERIAN THEOREM[1]

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1. **Introduction.** Let f be an L^1 function whose Fourier transform $\hat{f}$ is free of (real) zeros. We will refer to such a function f as a Wiener kernel, and write $f \in W$. Let s be an L^∞ function which is slowly oscillating:

$$s(x) - s(y) \to 0 \quad \text{as } x \to \infty \text{ and } x - y \to 0.$$

Finally suppose that

$$(f * s)(x) = \int_{-\infty}^{\infty} f(x - t)s(t)\, dt \to 0 \quad \text{as} \quad x \to \infty.$$

Then

$$s(x) \to 0 \quad \text{as } x \to \infty.$$

This is Pitt's form [**4**], [**5**] of Wiener's Tauberian theorem [**7**], [**8**].

The above Tauberian theorem is easily derived from a closure theorem, also due to Wiener (loc. cit.), which asserts that for any $f \in W$ the finite linear combinations of translates $f(x+\lambda)$, λ real, are dense in L^1. Thus by the continuous linear functionals test, Wiener's Tauberian theorem is a consequence of the following

THEOREM A. *For any Wiener kernel f, the equation*

$$f * g = 0, \qquad g \in L^\infty, \tag{1}$$

implies that $g = 0$.

It is also possible, as indicated by Beurling [**1**], to prove directly that Wiener's theorem is a consequence of Theorem A.

A heuristic proof of Theorem A goes as follows. By Fourier transformation, equation (1) becomes $\hat{f}\hat{g} = 0$. Thus since $\hat{f}$ is free of zeros one must have $\hat{g} = 0$, and hence $g = 0$. The only difficulty with this approach is that for arbitrary $g \in L^\infty$, the Fourier transform $\hat{g}$ is a (tempered) distribution, and the product $\hat{f}\hat{g}$ is not defined in the usual theory (cf. [**3**], however). In the present note we indicate how one can get around this problem by replacing f with a suitable testing function of rapid descent, that is, a function belonging to Schwartz's

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space S [6]. (It should be mentioned that Beurling has given several proofs of Theorem A [1], [2], and that the second one cited also employs a generalized Fourier transform of g.)

2. **Two simple lemmas.** Let ϕ be any testing function of rapid descent, and define functions ϕ_n by setting

$$\phi_n(x) = \frac{1}{n}\,\phi\left(\frac{x}{n}\right), \qquad n = 1, 2, \cdots. \tag{2}$$

LEMMA 1. *For $f \in L^1$ and ϕ_n as above,*

$$\|f * \phi_n - \hat{f}(0)\phi_n\| \to 0 \quad \text{as } n \to \infty.$$

PROOF. The norm in question is given by

$$\int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} f(t)\{\phi_n(x-t) - \phi_n(x)\}\, dt \right| \leqq \int_{-\infty}^{\infty} |f(t)|\, \rho_n(t)\, dt,$$

where

$$\rho_n(t) = \|\phi_n(x-t) - \phi_n(x)\| = \|\phi(y - t/n) - \phi(y)\|.$$

It is clear that $\rho_n(t) \to 0$ for every fixed t, while $\rho_n(t) \leqq 2\|\phi\|$. The lemma thus follows from Lebesgue's dominated convergence theorem.

LEMMA 2. *Suppose that u and v belong to L^1 and that $\|v\| < 1$. Then $\hat{u}/(1+\hat{v})$ is the Fourier transform of an L^1 function w.*

PROOF. Consider the series

$$u - u * v + u * v * v - \cdots. \tag{3}$$

Since

$$\|u * v^{*n}\| \leqq \|u\|\,\|v\|^n,$$

the sum of the norms of the terms in (3) is finite. It follows that the series converges in L^1 to a function w which has the desired Fourier transform.

3. **Proof of Theorem A.** Suppose that $f \in W$, and that $g \in L^\infty$ satisfies equation (1). We introduce a testing function ϕ of rapid descent whose Fourier transform $\hat{\phi}$ is equal to 1 on $[-1, 1]$ and equal to 0 outside $(-2, 2)$.

Defining ϕ_n by equation (2), Lemma 1 shows that we can choose an index p so large that

$$\|f * \phi_p - \hat{f}(0)\phi_p\| < |\hat{f}(0)|.$$

It is clear that $\hat{\phi}_p(x)=\hat{\phi}(px)=1$ for $|x|\leqq 1/p$; we also note that $\hat{\phi}_{2p}(x)=0$ for $|x|\geqq 1/p$.

We now set

$$u = \frac{1}{\hat{f}(0)}\,\phi_{2p},\quad v = \frac{1}{\hat{f}(0)}\,\{f*\phi_p - \hat{f}(0)\phi_p\}.$$

By Lemma 2 the quotient

$$\frac{\hat{u}}{1+\hat{v}} = \frac{\hat{\phi}_{2p}}{\hat{f}(0)+\hat{f}\hat{\phi}_p-\hat{f}(0)\hat{\phi}_p} = \frac{\hat{\phi}_{2p}}{\hat{f}}$$

is the Fourier transform of an L^1 function w. For this w we will have $w*f=\phi_{2p}$, hence, by equation (1),

$$\phi_{2p}*g = w*f*g = 0. \tag{4}$$

Since ϕ_{2p} is a testing function of rapid descent we can take Fourier transforms in (4) to obtain

$$\hat{\phi}_{2p}\hat{g} = 0. \tag{5}$$

Observing that the testing function $\hat{\phi}_{2p}$ is equal to 1 for $|x|\leqq 1/2p$, one derives from (5) that the distribution $\hat{g}$ is equal to 0 at least on the open interval $|x|<1/2p$.

So far we have only used the nonvanishing of $\hat{f}(0)$. However, equation (1) shows that the convolution of $f(x)e^{-icx}$ and $g(x)e^{-icx}$ is equal to 0 for every real number c, hence, by the preceding argument, the nonvanishing of $\hat{f}(c)$ implies that $\hat{g}$ vanishes in a neighborhood of the arbitrary point c. We conclude that $\hat{g}=0$ on $(-\infty,\ \infty)$ and, hence, $g=0$.

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