EIGENVALUES OF MODULUS 1

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In this paper B is a (complex) Banach space, T a bounded linear operator on B, and C the set of unimodular complex numbers with the usual topology. We shall prove the following:

THEOREM. *If B is separable and if $\|T^n\| \leqq M < \infty$ for all $n = 1, 2, \cdots$, then T cannot have an uncountable number of eigenvalues of modulus* 1.

We require two lemmas. Two members z and w of C are said to be *dependent* if there are integers j and k such that $z^j w^k = 1$; otherwise z and w are called *independent*. It is well known (see, for example, p. 150 of [1]) that if z and w are independent, the sequence $\{(z^n, w^n): n = 1, 2, \cdots\}$ is dense in $C \times C$.

LEMMA 1. *Let V be an uncountable subset of C. Then V contains an uncountable subset U with the property that any two distinct members of U are independent.*

PROOF. For w and z in C let $w \sim z$ iff w and z are dependent. It is easy to verify that $\sim$ is an equivalence relation. Let $\mathcal{P}$ be the partition of C induced by $\sim$. Observe that each member of $\mathcal{P}$ is denumerably infinite. Thus an uncountable set V must intersect uncountably many members of $\mathcal{P}$. We may take for U any set obtained by selecting exactly one member from each of the nonvoid members of $\{P \cap V: P \in \mathcal{P}\}$.

LEMMA 2. *Suppose $\|T^n\| \leqq M$, $n = 1, 2, \cdots$. Let λ_1 and λ_2 be independent eigenvalues of modulus* 1 *for T. Then, if x_1 and x_2 are eigenvectors of norm* 1 *for λ_1 and λ_2, respectively, $\|x_1 - x_2\| \geqq 2/(M+1)$.*

PROOF. Since λ_1 and λ_2 are independent, there is an increasing sequence $\{n_k\}$ of positive integers such that $(\lambda_1^{n_k}, \lambda_2^{n_k}) \to (-1, 1)$. Now let $n_k \to \infty$ in the relation

$$\|x - \lambda_1^{n_k} x\| \leqq \|x - y\| + \|y - \lambda_2^{n_k} y\| + \|\lambda_2^{n_k} y - \lambda_1^{n_k} x\|.$$

The left-hand side approaches $\|x + x\| = 2$. $\|y - \lambda_2^{n_k} y\| \to \|y - y\| = 0$. Also $\|\lambda_2^{n_k} y - \lambda_1^{n_k} x\| = \|T^{n_k} x - T^{n_k} y\| \leqq M\|x - y\|$. Thus

$$2 \leqq (1 + M)\|x - y\|,$$

yielding the conclusion of the lemma.

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PROOF OF THE THEOREM. Assume that B is separable and that $\|T^n\| \leqq M < \infty$ for $n=1, 2, \cdots$. Let V be the set of eigenvalues of T of modulus 1. Suppose V is uncountable. By Lemma 1, V contains an uncountable subset U, any two distinct members of which are independent. For each member of U select an eigenvector of norm 1 corresponding to that member, and let D denote the subset of B thus selected. D is, of course, uncountable. Let $S_x(r)$ denote the open sphere of radius r and center x. Any two distinct members of $\{S_x(1/(M+1)) : x \in D\}$ are disjoint by virtue of Lemma 2. This clearly contradicts the separability of B. Thus V is countable, and the theorem is proved.

Let $X = C^C$ with the product topology. A typical member of X is written as $\{x_\lambda\}_{\lambda \in C}$. Let ϕ be the map of X onto itself which multiplies the λth coordinate by λ; that is ϕ: $\{x_\lambda\}_{\lambda\in C} \to \{\lambda x_\lambda\}_{\lambda \in C}$. Let B be complex $C(X)$. Let $Tf = f \circ \phi$, $f \in C(X)$. Then $\|T\| = 1$, so $\|T^n\| \leqq 1$. If f is that member of $C(X)$ taking a point into its λth coordinate, f is an eigenfunction for λ. Thus all points in C are eigenvalues of T. Of course, B is not separable.

Even with B separable and $\|T\| = 1$, there may be an uncountable number of eigenvalues. Consider, for example, the shift operator S on complex l_2:

$$S(\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots) = (\lambda_2, \lambda_3, \cdots, \lambda_{n+1}, \cdots).$$

If $|\lambda| < 1$, $x = (\lambda, \lambda^2, \cdots, \lambda^n, \cdots)$ belongs to l_2 and $Sx = \lambda x$. $\|S\| = 1$, and S has for eigenvalues all complex numbers with modulus strictly less than 1. However, it is easily verified that S has no eigenvalues of modulus 1.

If T has an eigenvalue of modulus 1, then $\|T\| \geqq 1$. If $\|T^n\| \leqq M$, we have $1 \leqq \|T^n\| \leqq M$, so $1 \leqq \|T^n\|^{1/n} \leqq M^{1/n} \to 1$ as $n \to \infty$. Thus the spectral radius of T is 1. It is natural to wonder whether a T of spectral radius 1 on a separable Banach space B can have uncountably many eigenvalues of modulus 1. The following operator (suggested to the author by G. Kalisch) shows that it can indeed happen. Let $B = L_2(0, 1)$. Let M_0 and V_0 be defined by $(M_0 f)(x) = (\exp 2\pi i x) f(x)$, $(V_0 f)(x) = \int_0^x (\exp 2\pi i y) f(y)\, dy$. It is easily checked that, for each $\theta \in [0, 1)$, the characteristic function of the interval $(\theta, 1)$ is an eigenfunction for $M_0 - V_0$ corresponding to the eigenvalue $\exp 2\pi i\theta$. Showing that $M_0 - V_0$ has spectral radius 1 is not trivial, and the following elegant argument is also due to Kalisch. A direct computation shows that

$$V_0^2 = M_0 V_0 - V_0 M_0, \tag{1}$$

and it then follows by induction that

$$(2) \qquad (M_0 - V_0)^n = M_0^n - nV_0M_0^{n-1} \qquad \text{for } n \geqq 1.$$

Thus $\|(M_0 - V_0)^n\| \leqq 1 + 2\pi n$, so the spectral radius of $M_0 - V_0$ is 1. We carry this argument further. The spectrum of M_0 is C, so $(z - M_0)^{-1}$ exists for $|z| \neq 1$. Using (2), we see, by expanding $(z - M_0)^{-1}$ and $(z - M_0)^{-2}$ in negative powers of z, that

$$(3) \qquad (z - (M_0 - V_0))^{-1} = (z - M_0)^{-1} - V_0(z - M_0)^{-2}$$

holds for $|z| > 1$. It can be shown directly (using (1)) that (2) holds for $n = -1$. A simple induction argument (again using (1)) shows that (2) holds for all $n = -1, -2, \cdots$. We now expand $(z - M_0)^{-1}$ and $(z - M_0)^{-2}$ in positive powers of z to show that (3) holds for $|z| < 1$. Thus the spectrum of $M_0 - V_0$ is C, and is entirely point spectrum.

Bibliography

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