A CHARACTERIZATION OF OPERATORS MAPPING A CONE INTO ITS DUAL

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Introduction. Our point of departure is the following remark of Aronszajn and Smith, the n-dimensional case of a (real) Hilbert-space theorem contained in [1]. Let A be a real symmetric nonsingular matrix, with associated quadratic form (Ax, x). A necessary and sufficient condition that the elements of A^{-1} all be non-negative is that, to every $x \in R^n$ (R^n = the space of real n-tuples), there exists a corresponding $\bar{x}$ with the properties (a) $\bar{x}_i \geqq |x_i|$, $i = 1, \cdots, n$ and (b) $(A\bar{x}, \bar{x}) \leqq (Ax, x)$.

If we introduce the orthant $P_1 = \{x: x_i \geqq 0, i = 1, 2, \cdots, n\}$, then the remark states that a necessary and sufficient condition for $A^{-1}P_1 \subseteq P_1$ is that, to each x corresponds an $\bar{x}$ satisfying (a) $\bar{x} \pm x \in P_1$ and (b) $(A\bar{x}, \bar{x}) \leqq (Ax, x)$. We give below a modification of the Aronszajn and Smith theorem, obtained by a modification of their proof whose essentials are already contained in [1]. The purpose of the modification is to replace the positive orthant P_1 by an arbitrary convex cone P. Indeed, the use of reproducing kernels in [1] needlessly restricts considerations in n-space there to cones with n linearly independent generators. (With a suitable scalar product, and referred to a suitable basis, such a cone is essentially the non-negative orthant P_1 above.) By contrast, the theorem as formulated below exploits their considerations geometrically, without the use of reproducing kernels, extending its scope. In particular, the same remark holds for certain cones other than P_1 in n-space which have infinitely many generators, and for which the methods of [1] cannot apply directly. Throughout we use the notion of the dual P^* of a convex cone P: $P^* = \{x: (y, x) \geqq 0 \text{ all } y \in P\}$ or, more briefly, $P^* = \{x: (P, x) \geqq 0\}$.

Let V be a real Hilbert space whose scalar product is written (x, y) and let P be a closed convex cone in V. We consider a map of V onto itself by a nonsingular self-adjoint operator B (that is, $(Bx, y) = (x, By)$ for all x and y). The cone P is mapped into the set BP, which is easily seen to again be a cone. How is the dual $(BP)^*$ of this new cone related to P^*, the dual of P? The answer is evident:

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$$(BP)^* = \{x: (BP, x) \geqq 0\} = \{x: (P, Bx) \geqq 0\}$$
$$= \{x: Bx \in P^*\} = B^{-1}(P^*).$$

(That is, P^* transforms contragrediently.)

Let us say that a cone is *obtuse* if $P \supset P^*$, *acute* if $P \subset P^*$, and *self-dual* if both inclusions hold. For example, the positive orthant in $R^n = \{x: x_i \geqq 0, i = 1, \cdots, n\}$, is self-dual. The terminology is motivated by the fact that, in two dimensions, a cone is acute (obtuse) according to this definition if and only if its central angle is at most (at least) $\pi/2$. For later convenience we formulate here

LEMMA 0. *BP is obtuse iff $B^{-2}P^* \subset P$.*

PROOF. This is immediate from $(BP)^* = B^{-1}P^*$. We remark that the definition of P^* and, hence, of obtuseness depend on the choice of scalar product.

We next introduce the ordering on V induced by P: $x \geqq 0$ if and only if $x \in P$, and $x \geqq y$ if and only if $x - y \geqq 0$. The analogue of the condition (a) of the introduction for cones P other than the non-negative orthant is $\tilde{x} \geqq \pm x$, or, equivalently, $\tilde{x} \pm x \in P$. We observe that this implies $\tilde{x} \geqq 0$, or, equivalently, $\tilde{x} \in P$.

LEMMA 1. *A sufficient condition that a closed convex cone Q be obtuse with respect to a given scalar product is that, given $u \in V$, there exists a $\tilde{u}$ such that* (a) *$\tilde{u} \geqq \pm u$ (with respect to Q) and* (b) *$(\tilde{u}, \tilde{u}) \leqq (u, u)$.*

REMARK. It is sufficient for such $\tilde{u}$ to exist only for $u \in Q^*$, as the proof below shows.

PROOF. We must show $Q^* \subset Q$. Given any u and a corresponding $\tilde{u}$, form $u^+ = (\tilde{u} + u)/2$, $u^- = (\tilde{u} - u)/2$. Clearly, $u^+ \geqq 0$, $u^- \geqq 0$, $(u^+, u^-) = ((\tilde{u}, \tilde{u}) - (u, u))/4 \leqq 0$ and $\tilde{u} = u^+ + u^-$. Let it now be assumed that (c) $u \in Q^*$. We will establish the desired result by showing (a), (b) and (c) imply $u = \tilde{u}$, for we have already observed that (a) implies $\tilde{u} \in Q$. To begin with, $(u^+, u^-) \leqq 0$. Since u^+, u^- are in Q, while $u \in Q^*$, we have $(u, u^-) \geqq 0$. But $(u, u^-) = (u^+, u^-) - (u^-, u^-) \geqq 0$ implies, in virtue of the above inequalities, that $(u^+, u^-) = 0$ and $-(u^-, u^-) = 0$ since each is nonpositive and their sum non-negative. We conclude that $u^- = 0$, and so $\tilde{u} = u$. Since $\tilde{u} \in Q$, we obtain $u \in Q$, i.e., $Q^* \subset Q$.

LEMMA 2. *The sufficient conditions* (a) *and* (b) *stated in Lemma* 1 *are also necessary: given that $Q \supset Q^*$, then, for each $u \in V$, there exists a $\tilde{u}$ such that* (a) *$\tilde{u} \geqq \pm u$ and* (b) *$(\tilde{u}, \tilde{u}) \leqq (u, u)$.*

PROOF. Given that Q is obtuse, and given an arbitrary vector u, we must find the corresponding $\tilde{u}$. Let u_+ = orthogonal projection of u

on Q = closest element of Q to u; let u_- = orthogonal projection of $-u$ on $Q = -$(orthogonal projection of u on $-Q$).

We assert that $u_+ - u \in Q^*$ and $u_- + u \in Q^*$. In fact, the statement $u_+ - u \in Q^*$, together with $(u_+ - u, u_+) = 0$, is the analog, for cones, of the well-known orthogonality relation that characterizes projections of a subspace M: the vector u_+ of Q minimizing $\|u - v\|^2$ for all choice of v in Q must make $\|u - (u_+ + tv)\|^2 \geqq \|u - u_+\|^2$ for arbitrary $v \in Q$ and $t \geqq 0$. Thus $\|u - u_+\|^2 - 2t(u - u_+, v) + t^2\|v\|^2 \geqq \|u - u_+\|^2$ for all $t \geqq 0$, and, therefore, $(u_+ - u, v) \geqq 0$, i.e., $u_+ - u \in Q^*$. If we examine the case $v = u_+$ and $|t|$ small, we find, moreover, $(u - u_+, u_+) = 0$. Similarly, $u_- + u \in Q^*$ and $(u_- + u, u_-) = 0$. Thus, if we let $\tilde{u} = u_+ + u_-$, we have $\tilde{u} \pm u \in Q^* + Q \subset Q$, which is condition (a). We need only verify (b): $(\tilde{u}, \tilde{u}) \leqq (u, u)$. However, we recall $\|v \pm w\|^2 = \|v\|^2 + \|w\|^2$ if $(v, w) = 0$. In particular,

$$\left\|u_+ - \frac{u}{2}\right\|^2 = \left\|\frac{u_+}{2} + \frac{u_+ - u}{2}\right\|^2 = \left\|\frac{u}{2}\right\|^2,$$

$$\left\|u_- + \frac{u}{2}\right\|^2 = \left\|\frac{u_-}{2} + \frac{u_- + u}{2}\right\|^2 = \left\|\frac{u}{2}\right\|^2,$$

and, therefore, $\|\tilde{u}\|^2 = \|u_+ + u_-\|^2 = \|u_+ - u/2 + u_- + u/2\|^2 \leqq (\|u\|/2 + \|u\|/2)^2 = \|u\|^2$.

THEOREM. *A necessary and sufficient condition that a positive definite self-adjoint operator A satisfy $A^{-1}P^* \subset P$ is that, for any x, there exists an $\tilde{x}$ such that* (a) $\tilde{x} \geqq \pm x$ (*i.e.*, $\tilde{x} \pm x \in P$) *and* (b) $(A\tilde{x}, \tilde{x}) \leqq (Ax, x)$.

PROOF. Since $A = BB^*$ where B itself is self-adjoint, we may write $A^{-1}P^* \subset P$ in the form $B^{-1}P^* \subset BP$. Letting $Q = BP$, the inclusion relation of the theorem becomes the condition that Q be obtuse by Lemma 0. Let $Bx = u$, $B\tilde{x} = \tilde{x}$. The condition $\tilde{x} \geqq \pm x$ with respect to the ordering induced by P is equivalent to $\tilde{u} \geqq \pm u$ with respect to the ordering induced by $Q = BP$. Also, $(Ax, x) = (Bx, Bx) = (u, u)$ and $(A\tilde{x}, \tilde{x}) = (\tilde{u}, \tilde{u})$. Thus the theorem is transferable, under the mapping by B, to a necessary and sufficient condition that the cone Q be obtuse. The necessity and sufficiency are established in the two previous lemmas.

COROLLARY 1. *If $P = P^*$, the above is a necessary and sufficient condition that $A^{-1}P \subset P$. If the cone P is generated by n independent vectors, then $P = P^*$ for any choice of scalar product rendering these vectors orthogonal. The theorem then is applicable to those A which are positive definite and self-adjoint with respect to such a scalar product*; *in this* *form, the theorem reduces to the case discussed in* [1], *which considers, without further loss of generality, the case* $V=R^n$, $P=$ *positive orthant, and the usual scalar product. However, there are cones for which* $P=P^*$ *and yet are not orthants no matter what the choice of scalar product since they do not have a set of n generators* (*for example, the Lorentz or right circular cone,* $\{x\colon x_1\geqq(x_2^2+x_3^2+\cdots+x_n^2)^{1/2}\}$). *For these cones the theorem generalizes the result in* [1].

REMARKS. (1) In general, the theorem stated here is a modification rather than a generalization of that in [1], since $A^{-1}P^*\subset P$ neither implies nor is implied by $A^{-1}P\subset P$. However, it is interesting to observe that if P is *acute* ($P\subset P^*$), then $A^{-1}P^*\subset P\Rightarrow A^{-1}P^*\subset P^*$, while if P is *obtuse*, $A^{-1}P\subset P\Rightarrow A^{-1}P^*\subset P$. Thus (a) and (b) are sufficient for $A^{-1}P^*\subset P^*$ and necessary for $A^{-1}P\subset P$ in the acute and obtuse cases, respectively.

(2) A closed convex cone P in V with vertex 0 determines a like cone $P_M=M\cap P$ in any closed subspace M and the scalar product in V restricts to a scalar product in M. Thus we can introduce a relativized notion of obtuseness for P_M. It is easy to see that the relative dual of P_M contains $P^*\cap M$, and, therefore P_M is always obtuse if P is. The content of Lemmas 1 and 2 is that the obtuseness of P is equivalent to the relative obtuseness of P_M for those two-dimensional subspaces M spanned by vectors u_+ and u_- which arise by projecting an arbitrary u on P and $-P$. In particular, *a cone is obtuse if its two-dimensional sections with nonempty relative interior are all obtuse.* I know no direct proof of this.

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