

ON THE RICCI AND WEINGARTEN MAPS OF A HYPERSURFACE¹

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The purpose of this note is to prove a classical type relation between the Ricci map R^* and the Weingarten map L of a hypersurface in a flat Riemannian manifold. Indeed, if H is the mean curvature of the hypersurface, then $L^2 - HL + R^* = 0$. This can be viewed, equivalently, as a relation between the Ricci tensor and the second and third fundamental forms. Some obvious corollaries follow.

Let M be an n -dimensional C^∞ Riemannian manifold, let X and Y be vectors in M_m , the tangent space at a point m in M , and let $R(X, Y)$ be the skew-symmetric linear transformation valued curvature tensor determined by X and Y (see Helgason [2]). We say M is flat if $R \equiv 0$. The Ricci map R^* is the self-adjoint linear map $R^*(X) = \sum R(X, Z_i)Z_i$, where $i = 1, \dots, n$ (and this is the standard domain for sum indices in this note), and Z_1, \dots, Z_n are an orthonormal base of M_m . Thus $\text{Ric}(X, Y) = \langle R^*(X), Y \rangle$ gives the symmetric covariant Ricci tensor (this is the negative of the classical Ricci tensor), and $\text{Ric}(X, X)$ is the Ricci curvature of X . If R^* is a multiple of the identity map at every point of M , then M is called an Einstein manifold (see Goldberg [1, p. 38]). If $R^* \equiv 0$ on M , then M is Ricci flat. An eigenvector of R^* is called a Ricci vector.

We will further assume M is a C^∞ hypersurface in a flat C^∞ Riemannian manifold \bar{M} of dimension $(n+1)$ and N is a C^∞ unit normal vector field on M . Let D be the Riemannian covariant differentiation operator on \bar{M} . The Weingarten map L is the self-adjoint linear map on each tangent space M_m defined by $L(X) = D_X N$ (see Hicks [3]). The algebraic invariants of L are the imbedded differential geometric invariants of M ; in particular, the mean curvature $H = \sum k_i$, is the trace of L (k_i are the principal curvatures), the second mean curvature $J = \sum_{i < j} k_i k_j$, and the total curvature K is the determinant of L .

For the rest of this note M shall be as described in the above two paragraphs.

THEOREM. $L^2 - HL + R^* = 0$.

PROOF. We start with the Gauss-curvature equation (see Hicks

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[4]) which states that when X , Y , and Z are tangent to M at any point m in M , then

$$R(X, Y)Z = \langle L(Y), Z \rangle L(X) - \langle L(X), Z \rangle L(Y).$$

Let Z_1, \dots, Z_n be an orthonormal base of principal vectors at m with $L(Z_i) = k_i Z_i$.

Then

$$\begin{aligned} R^*(X) &= \sum R(X, Z_i)Z_i \\ &= \sum [\langle k_i Z_i, Z_i \rangle L(X) - \langle L(X), Z_i \rangle k_i Z_i] \\ &= HL(X) - \sum \langle L(X), L(Z_i) \rangle Z_i \\ &= HL(X) - \sum \langle L^2(X), Z_i \rangle Z_i \\ &= HL(X) - L^2(X), \end{aligned}$$

since L is self-adjoint.

COROLLARY 1. *Every principal vector X is a Ricci vector.*

COROLLARY 2. *Every surface in three space is an Einstein manifold.*

PROOF. If $n=2$, the characteristic equation for L is $L^2 - HL + KI = 0$, where I is the identity map, so $R^* = KI$.

COROLLARY 3. *If the Ricci curvature of every vector on M is nonpositive, then the second mean curvature is nonpositive on M .*

PROOF. Let X be a unit principal vector with $L(X) = kX$. Then $\text{Ric}(X, X) = \langle R^*(X), X \rangle = kH - k^2 \leq 0$. Thus, for each principal curvature k_i , $\sum_{j=1, j \neq i}^n k_i k_j \leq 0$, which implies $\sum_{i < j} k_i k_j = J \leq 0$.

COROLLARY 4. *On a minimal hypersurface the Ricci curvature of every vector is nonpositive and hence the second mean curvature is nonpositive.*

PROOF. In this case, $\text{Ric}(X, X) = \langle R^*(X), X \rangle = \langle -L^2(X), X \rangle = -\langle L(X), L(X) \rangle \leq 0$.

COROLLARY 5. *An Einstein hypersurface has at most two distinct principal curvatures.*

PROOF. If $R^* = bI$, then every principal curvature must satisfy the equation $k^2 - Hk + b = 0$.

COROLLARY 6. *A hypersurface is Ricci flat if and only if it is Einstein with total curvature zero. If $n=3$ and M is Ricci flat, then the second mean curvature is also zero so at points m on M that are not flat points ($L_m \neq 0$), the multiplicity of the nonzero principal curvature is unity.*

PROOF. If M is Ricci flat it is trivially Einstein and, since $L^2 - HL = 0$, one of the principal curvatures is zero.

If $R^* = bI$, then $L^2 - HL + bI = 0$. If $K = 0$, then there is a zero principal curvature and a unit principal vector X with $LX = 0$. Hence $bX = 0$ so $R^* = 0$.

In the case $n = 3$, the characteristic polynomial $L^3 - HL^2 + JL - KI = 0$ implies $JL = 0$, and since $L_m = 0$ implies $J(m) = 0$, we have $J \equiv 0$.

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ON PSEUDOMETRICS FOR GENERALIZED UNIFORM STRUCTURES

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In [1] Alfsen and Njåstad generalized the concept of a uniform structure \mathfrak{U} on a set S , replacing the intersection axiom for uniform structures by the weaker condition:

(0) Given subsets A_1, \dots, A_n of S and U_1, \dots, U_n in \mathfrak{U} , there exists U in \mathfrak{U} such that $U(A_i) \subseteq U_i(A_i)$ for $i = 1, \dots, n$. Our object is to characterize these structures in terms of pseudometrics.

Define a (generalized) *gage* on S to be a nonvoid family \mathfrak{G} of pseudometrics on $S \times S$ such that

(1) Every pseudometric uniformly continuous with respect to \mathfrak{G} belongs to \mathfrak{G} .

(2) If α and β belong to \mathfrak{G} and both α and β are totally bounded, then $\alpha \vee \beta$ belongs to \mathfrak{G} .

Note that if we delete the total boundedness condition in (2), then \mathfrak{G} is just a *gage* for a proper uniform structure [2], [3]. For β a pseudometric on $S \times S$, define $W_\beta = \beta^{-1}[0, 1)$.

THEOREM. *Given a gage \mathfrak{G} on S , define the class \mathfrak{U} of subsets U of $S \times S$ by the condition*

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