ON THE RICCI AND WEINGARTEN MAPS OF A HYPERSURFACE[1]

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The purpose of this note is to prove a classical type relation between the Ricci map R^* and the Weingarten map L of a hypersurface in a flat Riemannian manifold. Indeed, if H is the mean curvature of the hypersurface, then $L^2 - HL + R^* = 0$. This can be viewed, equivalently, as a relation between the Ricci tensor and the second and third fundamental forms. Some obvious corollaries follow.

Let M be an n-dimensional C^∞ Riemannian manifold, let X and Y be vectors in M_m, the tangent space at a point m in M, and let $R(X, Y)$ be the skew-symmetric linear transformation valued curvature tensor determined by X and Y (see Helgason [2]). We say M is flat if $R \equiv 0$. The *Ricci map* R^* is the self-adjoint linear map $R^*(X) = \Sigma R(X, Z_i)Z_i$, where $i = 1, \cdots, n$ (and this is the standard domain for sum indices in this note), and $Z_1, \cdots, Z_n$ are an orthonormal base of M_m. Thus $\mathrm{Ric}(X, Y) = \langle R^*(X), Y\rangle$ gives the symmetric covariant *Ricci tensor* (this is the negative of the classical Ricci tensor), and $\mathrm{Ric}(X, X)$ is the *Ricci curvature of* X. If R^* is a multiple of the identity map at every point of M, then M is called an *Einstein manifold* (see Goldberg [1, p. 38]). If $R^* \equiv 0$ on M, then M is *Ricci flat*. An eigenvector of R^* is called a *Ricci vector*.

We will further assume M is a C^∞ hypersurface in a flat C^∞ Riemannian manifold $\overline{M}$ of dimension $(n+1)$ and N is a C^∞ unit normal vector field on M. Let D be the Riemannian covariant differentiation operator on $\overline{M}$. The *Weingarten map* L is the self-adjoint linear map on each tangent space M_m defined by $L(X) = D_X N$ (see Hicks [3]). The algebraic invariants of L are the imbedded differential geometric invariants of M; in particular, the *mean curvature* $H = \Sigma k_i$ is the trace of L (k_i are the principal curvatures), the *second mean curvature* $J = \sum_{i<j} k_i k_j$, and the *total curvature* K is the determinant of L.

For the rest of this note M shall be as described in the above two paragraphs.

THEOREM. $L^2 - HL + R^* = 0$.

PROOF. We start with the Gauss-curvature equation (see Hicks

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[4]) which states that when X, Y, and Z are tangent to M at any point m in M, then

$$R(X, Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y).$$

Let $Z_1, \cdots, Z_n$ be an orthonormal base of principal vectors at m with $L(Z_i) = k_i Z_i$.
Then

$$\begin{aligned} R^*(X) &= \Sigma R(X, Z_i)Z_i \\ &= \Sigma[\langle k_i Z_i, Z_i\rangle L(X) - \langle L(X), Z_i\rangle k_i Z_i] \\ &= HL(X) - \Sigma\langle L(X), L(Z_i)\rangle Z_i \\ &= HL(X) - \Sigma\langle L^2(X), Z_i\rangle Z_i \\ &= HL(X) - L^2(X), \end{aligned}$$

since L is self-adjoint.

COROLLARY 1. *Every principal vector X is a Ricci vector.*

COROLLARY 2. *Every surface in three space is an Einstein manifold.*

PROOF. If $n=2$, the characteristic equation for L is $L^2 - HL + KI = 0$, where I is the identity map, so $R^* = KI$.

COROLLARY 3. *If the Ricci curvature of every vector on M is nonpositive, then the second mean curvature is nonpositive on M.*

PROOF. Let X be a unit principal vector with $L(X) = kX$. Then $\mathrm{Ric}(X, X) = \langle R^*(X), X\rangle = kH - k^2 \leqq 0$. Thus, for each principal curvature k_i, $\sum_{j=1, j\neq i}^{n} k_i k_j \leqq 0$, which implies $\sum_{i<j} k_i k_j = J \leqq 0$.

COROLLARY 4. *On a minimal hypersurface the Ricci curvature of every vector is nonpositive and hence the second mean curvature is nonpositive.*

PROOF. In this case, $\mathrm{Ric}(X, X) = \langle R^*(X), X\rangle = \langle -L^2(X), X\rangle = -\langle L(X), L(X)\rangle \leqq 0$.

COROLLARY 5. *An Einstein hypersurface has at most two distinct principal curvatures.*

PROOF. If $R^* = bI$, then every principal curvature must satisfy the equation $k^2 - Hk + b = 0$.

COROLLARY 6. *A hypersurface is Ricci flat if and only if it is Einstein with total curvature zero. If $n=3$ and M is Ricci flat, then the second mean curvature is also zero so at points m on M that are not flat points ($L_m \neq 0$), the multiplicity of the nonzero principal curvature is unity.*

PROOF. If M is Ricci flat it is trivially Einstein and, since $L^2-HL=0$, one of the principal curvatures is zero.

If $R^*=bI$, then $L^2-HL+bI=0$. If $K=0$, then there is a zero principal curvature and a unit principal vector X with $LX=0$. Hence $bX=0$ so $R^*=0$.

In the case $n=3$, the characteristic polynomial $L^3-HL^2+JL-KI=0$ implies $JL=0$, and since $L_m=0$ implies $J(m)=0$, we have $J\equiv 0$.

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ON PSEUDOMETRICS FOR GENERALIZED UNIFORM STRUCTURES

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In [1] Alfsen and Njåstad generalized the concept of a uniform structure $\mathcal{U}$ on a set S, replacing the intersection axiom for uniform structures by the weaker condition:

(0) Given subsets $A_1, \cdots, A_n$ of S and $U_1, \cdots, U_n$ in $\mathcal{U}$, there exists U in $\mathcal{U}$ such that $U(A_i)\subseteq U_i(A_i)$ for $i=1, \cdots, n$. Our object is to characterize these structures in terms of pseudometrics.

Define a (generalized) *gage* on S to be a nonvoid family $\mathcal{G}$ of pseudometrics on $S\times S$ such that

(1) Every pseudometric uniformly continuous with respect to $\mathcal{G}$ belongs to $\mathcal{G}$.

(2) If α and β belong to $\mathcal{G}$ and both α and β are totally bounded, then $\alpha\vee\beta$ belongs to $\mathcal{G}$.

Note that if we delete the total boundedness condition in (2), then $\mathcal{G}$ is just a gage for a proper uniform structure [2], [3]. For β a pseudometric on $S\times S$, define $W_\beta=\beta^{-1}[0, 1)$.

THEOREM. *Given a gage $\mathcal{G}$ on S, define the class $\mathcal{U}$ of subsets U of $S\times S$ by the condition*

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