AN L_p THEORY FOR A MARKOV PROCESS WITH A SUB-INVARIANT MEASURE[1]

S. R. FOGUEL

Let a Markov process be given by its transition probability. If the process has a sub-invariant measure then it defines an operator (or a semigroup of operators) on L_p, $1 \leqq p \leqq \infty$. Using this representation we study:

1. The behaviour of $P^n(x, A)$ as $n \to \infty$.
2. Continuity and differentiability properties of $P_t(x, A)$.

(The notation is explained below.)

Kendall in [6] and [7] uses a similar method in the case where the state space is discrete and the process irreducible. Instead of irreducibility we assume a "Doeblin condition" and we do not restrict the state space. In [1], [2] and [3] the same problems were studied. In this paper we do not assume that the measure is invariant but only sub-invariant (this idea was employed in Kendall [6], [7]). Also we do not assume that the process is "honest"; we have $P(x, X) \leqq 1$ instead of equality.

1. **Definitions and notation.** Let (X, Σ, μ) be a measure space (the State Space) where $\mu \geqq 0$ but is not necessarily finite. Let $P(x, A)$ be a transition probability:

(a) *$P(x, A)$ is defined for $x \in X$ and $A \in \Sigma$ and $0 \leqq P(x, A) \leqq 1$.*

(b) *For a fixed x the function $P(x, \cdot)$ is a measure on Σ.*

(c) *For a fixed $A \in \Sigma$ the function $P(\cdot, A)$ is measurable.*

The measure μ is assumed to satisfy

$$\mu(A) \geqq \int P(x, A)\, \mu(dx). \tag{1.1}$$

(The measure μ is sub-invariant.)

Let

$$(Pf)(x) = \int f(y)\, P(x, dy). \tag{1.2}$$

Now if $f = \sum c_i I(A_i)$, where $I(A)$ denotes the characteristic function of A, and $\mu(A_i) < \infty$, then:

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$$\int |(Pf)(x)|\,\mu(dx) \leqq \sum |c_i| \int P(x, A_i)\,\mu(dx) \leqq \sum |c_i|\mu(A_i)$$

by equation (1.1).

The above argument shows that, on $L_1(X, \Sigma, \mu)$,

$$\|P\|_1 \leqq 1.$$

Since, clearly, $\|P\|_\infty \leqq 1$ we can use the Riesz Convexity Theorem to conclude:

For every $1 \leqq p \leqq \infty$ the operator P on $L_p(X, \Sigma, \mu)$ has norm less than or equal to one.

For our first problem the case $p=2$ will be important. We shall use the following theorems on L_2.

Let

$$H_0 = \{f \mid f \in L_2 \textit{ and } \text{weak lim } P^n f = 0\}.$$

$$H_1 = H_0^{\perp}.$$

$$K = \{f \mid f \in L_2 \textit{ and } \|P^n f\| = \|P^{*n} f\| = \|f\| \textit{ for all } n\}.$$

THEOREM 1.1. *The sets H_0, H_1 and K are subspaces invariant under P and P^*. The restriction of P to K is unitary and $H_1 \subset K$. The subspace K is of the form $L_2(X, \Sigma_1, \mu)$ where Σ_1 is a σ-subfield of Σ. If $\sigma \in \Sigma_1$ then $PI(\sigma) = I(\tau)$ where $\tau \in \Sigma_1$ and P is an automorphism of Σ_1.*

PROOF. This theorem follows from Theorem 1.1 of [5] and Theorem 2 of [4]. We have only to show that P^* satisfies Condition 3 of Definition 1 of [4]. Let $f \in L_2$ and $f \leqq c$. Let A be any set of finite measure, then

$$\int_A P^* f\,d\mu = \int f P(I(A))\,d\mu \leqq c \int P(x, A)\mu(dx) \leqq c\mu(A).$$

Thus

$$P^* f \leqq c \text{ a.e.}$$

LEMMA 1.2. *Let H be a Hilbert space and T an operator on H with $\|T\| \leqq 1$. For any $x \in H$:*

$$\text{weak lim } T^n x = 0 \qquad \textit{if } (T^{kn}x, x) \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{for some } k.$$

((x, y) denotes the inner product in H.)

PROOF. If $((T^k)^n x, x) \to 0$, then by Theorem 3.1 of [5], weak $\lim_{n\to\infty} (T^k)^n x = 0$. But then also

$$\text{weak} \lim_{n\to\infty} T^d T^{kn} x = 0, \qquad d = 0, 1, \cdots, k-1.$$

REMARK. If λ is a finite positive measure on $[0, 2\pi)$ and c_n are its Fourier coefficients, then the lemma implies: $c_{kn}\to 0$ *iff* $c_n \to 0$.

2. **Convergence of** $P_n(x, A)$ **as** $n\to\infty$. Let $P_n(x, A)$ be defined as usual; then

$$(P^n f)(x) = \int f(y)\, P^n(x, dy). \tag{2.1}$$

THEOREM 2.1. *Let A be a set of finite measure. If, for some integer k, the sequence $P^{kn}(x, A)$ converges in measure, on A, to zero, then $P^n(x, A)$ converges in measure to zero, on every set of finite measure.*

PROOF. Our assumption implies

$$\int_A P^{kn}(x, A)\, \mu(dx) \to 0.$$

Thus, by Lemma 1.2, the sequences $P^n(x, A) = P^n(I(A))$ converge weakly, in the L_2 sense, to zero. Since $P^n(x, A) \geqq 0$ this is the same as:

The sequence $P^n(x, A)$ converges in measure, on every set of finite measure, to zero.

If one studies the transition probability on (X, Σ_1, μ), then, since P is an automorphism on Σ_1, we have:

(d) *For any $A \in \Sigma_1$ there exist two sets B and C in Σ_1 such that*

$$P(x, A) = 1 \quad \text{if } x \in B, \qquad P(x, A) = 0 \quad \text{if } x \in' B \text{ a.e.}$$
$$P(x, C) = 1 \quad \text{if } x \in A, \qquad P(x, C) = 0 \quad \text{if } x \in' A \text{ a.e.}$$

The process on (X, Σ_1, μ) is a *Deterministic Process*: $P^n(x, A)$ is either zero or one, a.e., for every $x \in X$ and $A \in \Sigma_1$.

Let

$$X_0 = \{x \mid P(x, X) < 1\}.$$

LEMMA 2.2. *If $B \subset X_0$ and $A \in \Sigma_1$ then $\mu(B \cap A) = 0$.*

PROOF. By condition (d),

$$I(A) = P(x, C).$$

Thus, if $x \in A$,

$$P(x, X) \geqq P(x, C) = 1 \text{ a.e.} \quad \text{or} \quad \text{a.e. } x \in' B.$$

THEOREM 2.3. *Let B be a set of finite measure such that if $A \in \Sigma_1$ then $\mu(B \cap A) = 0$. Then* lim *in measure of* $P^n(x, B) = 0$ *on every set of finite measure.*

PROOF. Since $I(B)$ is orthogonal to K then, by Theorem 1.1, $I(B) \in H_1^{\perp} = H_0$.

We shall assume in the rest of this section that a "Doeblin Condition" holds, namely: there exists a positive measure λ, *on* Σ, *and an* $\epsilon > 0$ *such that*:

1. *If* $\mu(A) < \infty$ *then* $\lambda(A) < \infty$.
2. *If* $\lambda(A) < \epsilon$ *then for some* n, $P^n(x, A) < 1$ *for all* $x \in X$.

This kind of condition was used in [3]. If the state space X is discrete, then by taking λ to be the number of points in the set and $\epsilon = 1/2$ we see that the condition is trivially satisfied.

LEMMA 2.4. *The* σ*-field* Σ_1 *is generated by a collection of disjoint sets* $\{\sigma_\alpha\}$.

PROOF. It is enough to show that if $A \in \Sigma_1$ then it contains an atom of Σ_1. Now if A contains k disjoint sets of Σ_1 then $\epsilon \leqq \lambda(A)/k$ since every set in Σ_1 must have a λ-measure greater than ϵ. Thus the number of sets contained in A is smaller than $\lambda(A)/\epsilon$.

The operator P permutes the sets σ_α. Let us write $P^n\sigma_\alpha$ for the set whose characteristic function is $P^n(I(\sigma_\alpha))$. For each σ_α there are two possibilities:

1. The set σ_α is cyclic: for some $k(\alpha)$, $P^{k(\alpha)}\sigma_\alpha = \sigma_\alpha$.
2. The sets $P^n\sigma_\alpha$ are disjoint.

(We used here the fact that

$$\mu(P^n\sigma_\alpha \cap P^k\sigma_\alpha) = \mu(P^{n-k}\sigma_\alpha \cap \sigma_\alpha)$$

which holds since P is unitary on K.)

THEOREM 2.5. *Let* $A \in \Sigma$.

(a) *If* $A \cap \sigma_\alpha = \varnothing$ *for every* α *then the sequence* $P^n(x, A)$ *tends, in measure, to zero, on every set of finite measure.*

(b) *If* $A \subset \sigma_\alpha$ *where* σ_α *is not cyclic then the sequence* $P^n(x, A)$ *tends, in measure, to zero, on every set of finite measure.*

(c) *If* $A \subset \sigma_\alpha$ *and* $P^k\sigma_\alpha = \sigma_\alpha$ *and* B *is a set of finite measure then*:

$$\lim_{n\to\infty} \int_B P^{nk+d}(x, A)\,\mu(dx) = \mu(\sigma_\alpha)^{-1}\mu(A)\mu(B \cap P^d\sigma_\alpha),$$
$$d = 0, 1, \cdots, k-1.$$

PROOF. Case (a) was proved in Theorem 2.3. Case (b): Since the sets $P^n(\sigma_\alpha)$ are disjoint and

$$P^n(x, A) \leqq P^n(x, \sigma_\alpha) = I(P^n(\sigma_\alpha)),$$

the sequence $P^n(x, A)$ tends weakly, in the sense of L_2, to zero, which implies (b). Case (c): Note that

$$I(A) = \frac{\mu(A)}{\mu(\sigma_\alpha)} I(\sigma_\alpha) + f,$$

where f has its support in σ_α and is orthogonal (in L_2) to $I(\sigma_\alpha)$. Thus f is orthogonal to K and weak lim $P^n f = 0$ by Theorem 1.1. On the other hand,

$$P^{nk+d} I(\sigma_\alpha) = I(P^d \sigma_\alpha).$$

Thus

$$\int_B P^{nk+d}(x, A)\, \mu(dx) = \frac{\mu(A)}{\mu(\sigma_\alpha)} \mu(B \cap P^d \sigma_\alpha) + \int_B P^{nk+d} f\, d\mu$$

and

$$\int_B P^{nk+d} f\, d\mu \to 0.$$

3. **Markov process with continuous time parameter.** Let the transition probability be $P_t(x, A)$, $0 \leqq t$, where $P_t(x, A)$ satisfies conditions (a), (b) and (c) of the introduction and the Chapman-Kolmogoroff Equation:

$$P_{t+s}(x, A) = \int P_t(x, dy) P_s(y, A).$$

We shall replace equation (1.1) by

$$\mu(A) \geqq \int P_t(x, A)\, \mu(dx), \qquad t \geqq 0. \tag{3.1}$$

It is easily seen that the transition probability will define a semigroup of contractions on L_p, $1 \leqq p \leqq \infty$, by

$$(P_t f)(x) = \int f(y)\, P_t(x, dy).$$

We shall study in this section continuity and differentiability properties of the semigroup P_t.

Let us assume that $P_0(x, A) = I(A)(x)$.

If $\mu(A) < \infty$ and $P_t I(A) = P_t(x, A)$ is continuous in the L_p topology for $1 \leqq p < \infty$, then

$$\int_A P_t(x, A)\, \mu(dx) \to \mu(A), \tag{3.2}$$

since $I(A) \in L_q$ where $1/p+1/q=1$. Conversely, if (3.2) holds then

$$\|P_t(x, A) - I(A)\|_p^p = \int |P_t(x, A) - I(A)|^p \, d\mu$$
$$\leqq \int |P_t(x, A) - I(A)| \, d\mu,$$

since $0 \leqq P_t(x, A) \leqq 1$. Thus

$$\|P_t(x, A) - I(A)\|_p^p \leqq \int_A (1 - P_t(x, A)) \, d\mu + \int_{X-A} P_t(x, A) \, d\mu$$
$$= \mu(A) - \int_A P_t(x, A) \, d\mu + \int_{X-A} P_t(x, A) \, d\mu.$$

Now

$$\int_A P_t(x, A) \, d\mu + \int_{X-A} P_t(x, A) \, d\mu = \int_X P_t(x, A) \, d\mu \leqq \mu(A).$$

Thus

$$\|P_t(x, A) - I(A)\|_p^p \leqq 2\left(\mu(A) - \int_A P_t(x, A) \, \mu(dx)\right). \tag{3.3}$$

We shall assume that (3.2) *holds for every set* $A \in \Sigma$ *with* $\mu(A) < \infty$, *and thus the semigroup* P_t *is strongly continuous for* $1 \leqq p < \infty$ *and* $t \geqq 0$.

Note that if the process was given by the functions x_t then

$$\int_A P_t(x, A) \, \mu(dx) = \text{Probability}(x_t \in A \cap x_0 \in A).$$

If $P_t(x, A)$ is strongly differentiable for $t=0$ in the L_p sense $1 \leqq p < \infty$, then

$$\frac{1}{t}\left(\mu(A) - \int_A P_t(x, A) \, \mu(dx)\right) = \frac{1}{t}\int_A (I(A) - P_t(x, A)) \, \mu(dx)$$

would be bounded since $I(A) \in L_q$ $(1/p+1/q=1)$.

If one assumes that

$$t^{-p}\left[\mu(A) - \int_A P_t(x, A) \, \mu(dx)\right], \qquad p > 1,$$

is bounded, then, by (3.3), the quotient

$$t^{-1}(P_t(x, A) - I(A))$$

is bounded in the L_p norm. But L_p is a reflexive space and boundedness implies differentiability. In this case it is easy to see that

$$\frac{d}{dt}\int_A P_t(x, A)\,\mu(dx)\,\Big|_{t=0} = 0,$$

$$\frac{d}{dt}\int_B P_t(x, A)\,\mu(dx)\,\Big|_{t=0} = 0 \qquad \text{if } B \subset X - A.$$

We do not get any information on $(d/dt)\int_B P_t(x, A)\,\mu(dx)$, $B \subset A$.

DEFINITION. *Let $D^* \subset L_\infty(X, \Sigma, \mu) = L_1^*$ be the set*

$$\{f \mid f \in L_\infty \textit{ and } \lim \|P_t^* f - f\|_\infty = 0\}.$$

THEOREM 3.1. *The set D^* is a closed subspace of L_∞, invariant under P_t^*. For every $g \in L_1$,*

$$\|g\|_1 = \sup\left\{\left|\int g f\, d\mu\right| \,\Big|\, f \in D^* \textit{ and } \|f\|_\infty = 1\right\}.$$

This theorem is a consequence of Theorems 2.1, 2.2 and 3.1 of [8].

Throughout the rest of this paper A will denote a fixed set of finite measure such that

$$t^{-1}\left(\mu(A) - \int_A P_t(x, A)\,\mu(dx)\right)$$

is bounded.

THEOREM 3.2. *If $f \in D^*$, then the function $\int f(x) P_t(x, A)\,\mu(dx)$ is differentiable for every $t \geqq 0$.*

PROOF. Since $P_t^* f \in D^*$ it is enough to prove differentiability at $t = 0$. Now, by semigroup theory, the function $P_t^* g$ is strongly differentiable on a dense subset of D^*. On the other hand, for any h with $\|h\|_\infty \leqq 1$,

$$t^{-1}\left|\int h(x) P_t(x, A)\,\mu(dx) - \int_A h(x)\,\mu(dx)\right| \leqq t^{-1}\left(\mu(A) - \int_A P_t(x, A)\,\mu(dx)\right).$$

Thus the quotient

$$t^{-1}\left(\int h(x) P_t(x, A)\,\mu(dx) - \int_A h(x)\,\mu(dx)\right)$$

is uniformly bounded on $\|h\|_\infty \leqq 1$ and tends to a limit on a dense subset on D^*. Thus the limit exists for every $f \in D^*$.

Let $\Gamma \subset \Sigma$ be such that every set in Γ has a finite measure, and every set of finite measure, in Σ, is a countable union of sets in Γ.

Let $f \in L_\infty$:

$$\|P_t^* f - f\| = \sup\left\{\mu(B)^{-1}\left|\int_B (P_t^* f - f)\, d\mu\right| \,\Big|\, B \in \Gamma\right\}$$
$$= \sup\left\{\mu(B)^{-1}\left|\int f(x)(P_t(x, B) - I(B))\,\mu(dx)\right| \,\Big|\, B \in \Gamma\right\}.$$

In particular, $I(C) \in D^*$ if

$$\sup\left\{\mu(B)^{-1}\int_C P_t(x, B)\mu(dx) \,\Big|\, B \in \Gamma \text{ and } B \cap C = \varnothing\right\} \to 0, \qquad t \to 0,$$

and

$$\sup\left\{1 - \mu(B)^{-1}\int_C P_t(x, B)\mu(dx) \,\Big|\, B \in \Gamma \text{ and } B \subset C\right\} \to 0, \qquad t \to 0.$$

For a discreet process this implies $P_t(i, j)$ is differentiable for $t \geqq 0$ if

$$t^{-1}(1 - P_t(j, j))$$

is bounded and

$$\sup\{\mu_k^{-1}\mu_i P_t(i, k) \mid k, k \neq i\} \to 0, \qquad t \to 0,$$

where μ_k is the sub-invariant measure.

For our last result we shall need the notation of [3]. Let us add the assumptions:

$$P(x, X) = 1 \qquad \textit{for all } x \in X,$$

$$\mu(A) = \int P_t(x, A)\mu(dx).$$

One can define a measure space (Ω, T, ν) and a class of measurable functions x_t on Ω to X, where $t \geqq 0$, such that: If $T(t)$ is given by

$$T(t)f(x_0(\omega)) = f(x_t(\omega)),$$

E_0 is the self-adjoint projection, in $L_2(\Omega, T, \nu)$, on $L_2(\Omega, x_0^{-1}(\Sigma), \nu)$, then:

$$\mu(A) = \nu(\overset{-1}{x_0}(A)),$$

$$E_0T(t)I(x_0 \in A)(\omega) = P_t(x_0(\omega), A),$$

$$E_0T(t)f(x_0(\cdot))(\omega) = \int f(y)\, P_t(x_0(\omega), dy).$$

THEOREM 3.3. *Let* $f \in L_1(\mu) \cap L_\infty(\mu)$ *be such that*

$$\|f(x_t(\omega)) - f(x_0(\omega))\|_\infty \to 0, \qquad t \to 0;$$

then

$$\int f(y) P_t(y, A)\, \mu(dy)$$

is differentiable for $t \geqq 0$.

PROOF. It is enough to show that $f \in D^*$. Now for every $g \in L_1(\nu) \cap L_\infty(\nu)$,

$$(E_0T(t))^*g = T(t)^{-1}E(t)g,$$

where $E(t)$ is the self-adjoint projection on $L_2(\Omega, x_t^{-1}(\Sigma), \nu)$ (see Theorem 1.1 of [2]). Thus

$$\|(E_0T(t))^*f(x_0(\omega)) - f(x_0(\omega))\|_\infty = \|T(t)^{-1}(E(t)f(x_0(\omega)) - T(t)f(x_0(\omega))\|_\infty.$$

But $T(t)^{-1}$ is an isometry on L_∞; thus

$$\begin{aligned}\|(E_0T(t))^*f(x_0(\omega)) - f(x_0(\omega))\|_\infty &= \|E(t)f(x_0(\omega)) - T(t)f(x_0(\omega))\|_\infty \\ &= \|E(t)(f(x_0(\omega)) - T(t)f(x_0(\omega)))\|_\infty \leqq \|f(x_0(\omega)) - f(x_t(\omega))\|_\infty.\end{aligned}$$

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NORTHWESTERN UNIVERSITY AND
HEBREW UNIVERSITY, JERUSALEM, ISRAEL