A GENERALIZED KOSZUL COMPLEX. III.
A REMARK ON GENERIC ACYCLICITY

DAVID A. BUCHSBAUM AND DOCK SANG RIM

1. Introduction. Let $R$ be a commutative ring, and $S = R[X_{i1}, \ldots, X_{mn}]$ be the polynomial ring in $mn$ indeterminates over $R$. If we denote by $g(m, n) : S^m \to S^n$ the map whose matrix is $(X_{ij})$, we may consider the (generalized) Koszul complexes $K(\Lambda^p g(m, n))$ for $1 \leq p \leq n$ [2], [3]. One of the first things we shall show is that these complexes are acyclic, i.e. $H_i(K(\Lambda^p g(m, n))) = 0$ for $i > 0$. We will do this by using a result in a recent paper by Northcott [4], which says that if $R$ is noetherian, then the ideal, $I$, generated by the subdeterminants of order $n$ of the matrix $(X_{ij})$ is perfect, and $\text{depth}(I; S) = m - n + 1$.

Because the complexes $K(\Lambda^p g(m, n))$ are acyclic, we say that the generalized Koszul complex is \textit{generically acyclic}. This generic acyclicity enables us to interpret the homology and cohomology groups of the generalized Koszul complex as $\text{Tor}$ and $\text{Ext}$ over suitable rings, and also expresses these groups as the homology (and cohomology) of the classical Koszul complex. As a result, we are able to give easy proofs of the rigidity of these complexes [1], and also prove an assertion [2] concerning the annihilator of their homology groups.

Throughout this note, all rings are commutative and have an identity element.

2. Generic acyclicity. As in §1, we let $S = R[X_{i1}, \ldots, X_{mn}]$, where $X_{ij}$ are indeterminates over $R$, $1 \leq i \leq m$, $1 \leq j \leq n$ ($m \geq n$), and $g(m, n) : S^m \to S^n$ is the map whose matrix is $(X_{ij})$. The ideal $I(m, n)$ (or $I$) generated by the subdeterminants of $(X_{ij})$ of order $n$ is simply the image of the map $\Lambda^n g(m, n) : \Lambda^n S^m \to \Lambda^n S^n$.

\textbf{Theorem 2.1.} For each $p$ ($1 \leq p \leq n$), the complex $K(\Lambda^p g(m, n))$ [3, §1] has trivial homology in positive dimensions. Thus $K(\Lambda^p g(m, n))$ is a free resolution of $\text{Coker} \ \Lambda^p g(m, n)$.

\textbf{Proof.} We know, by Northcott's result quoted in §1, that, in case $R$ is noetherian, we have $\text{depth}(I; S) = m - n + 1$ [3, §2]. Thus, if $R$ is noetherian, the theorem follows immediately from Corollary 2.6 of [3]. In case $R$ is not noetherian, we observe that $R = \lim_{\to} R_a$ where $R_a$ is noetherian, and hence $S = \lim_{\to} S_a$, where $S_a = R_a[X_{i1}, \ldots, X_{mn}]$.

Received by the editors January 20, 1964.

This work was done with the partial support of NSF grant GP-218.
Therefore, $K(\Lambda^p g)(m,n), S) = \lim K(\Lambda^p g)(m,n), S)$ and we see that $K(\Lambda^p g)(m,n), S)$ is acyclic.

**Corollary 2.2.** Given a map $f: R^n \to R^m$ with matrix $(a_{ij})$ and a finitely generated $R$-module $E$, we have

$$H_s\left(\Lambda^p f, E\right) = \text{Tor}_s^R \left(\text{Coker} \Lambda^p g(m,n), S \otimes_R E\right)$$

$$H^s\left(\Lambda^p f, E\right) = \text{Ext}_s^R \left(\text{Coker} \Lambda^p g(m,n), \text{Hom}_R(S, E)\right)$$

where $S = R[X_{11}, \cdots, X_{mm}]$ and $S = S/I(X(f))$, with $I(X(f)) = (X_{11} - a_{11}, \cdots, X_{mn} - a_{mn})$. (For definitions of $H_s$ and $H^s$, see [3, §1].)

**Proof.** This follows immediately from the fact that $K(\Lambda^p g)(m,n)$ is a free resolution of $\text{Coker} \Lambda^p g(m,n)$ and that $P(A^p/) \otimes_R P = (P: (\Lambda^p g(m,n)) \otimes_R S) \otimes_R E = K(\Lambda^p g(m,n)) \otimes_R (S \otimes_R E)$ and

$$\text{Hom}_R \left(K(\Lambda^p f, E\right) = \text{Hom}_R \left(K(\Lambda^p g(m,n) \otimes_R S, E\right)$$

$$= \text{Hom}_R \left(K(\Lambda^p g(m,n), \text{Hom}_R(S, E)\right).$$

**Corollary 2.3.** $H(\Lambda^p f, \cdot)$ is annihilated by $\text{Ann} \text{Coker} \Lambda^p f$ (cf. [2]).

As another application we obtain the rigidity of the complex $K(\Lambda^p f)$.

**Theorem 2.4.** Let $f$ and $S$ be as in 2.2, with $R$ noetherian. If $X(f): S^{mn} \to S$ is the map whose $mn \times 1$ matrix is $(X_{11} - a_{11}, \cdots, X_{mn} - a_{mn})$, then $H_s(\Lambda^p f, E) = H_s(X(f), E \otimes_R \text{Coker} \Lambda^p g(m,n))$. In particular, $H_s(\Lambda^p f, E) = 0$ implies $H_t(\Lambda^p f, E) = 0$ for all $t \geq s$.

**Proof.** $K(\Lambda^p g)(m,n)$ is an acyclic resolution of $\text{Coker} \Lambda^p g(m,n)$, regardless of the coefficient ring $R$. In particular, this is so for $R/a$ for any ideal $a$ in $R$, i.e., $K(\Lambda^p g(m,n)) \otimes_R R/a$ is acyclic for any ideal $a$. Thus $\text{Tor}_s^R(\text{Coker} \Lambda^p g(m,n), R/a) = 0$ for all ideals $a$, since $K(\Lambda^p g(m,n))$ is $S$-free and, hence, an $R$-free resolution of $\text{Coker} \Lambda^p g(m,n)$. Therefore $\text{Coker} \Lambda^p g(m,n)$ is flat as an $R$-module.

Now the standard Koszul complex $K(X(f))$ is an $S$-free resolution of $\text{Coker} X(f) = S$, since $I(X(f))$ is generated by an $S$-sequence (as can be seen by linear substitution). Since $S = R$ as an $R$-module,
K\langle X(f) \rangle \otimes_R E is an acyclic resolution of \Sigma \otimes_R E, and each term of K\langle X(f) \rangle \otimes_R E is a direct sum of copies of S \otimes_R E. Now

\text{Tor}_i^S \left( \text{Coker } \Lambda g(m, n), S \otimes_R E \right) = \text{Tor}_i^R \left( \text{Coker } \Lambda g(m, n), E \right) = 0,

for all i > 0, since Coker \Lambda g(m, n) is R-flat. Therefore,

\begin{align*}
H_* \left( X(f), E \otimes_R \text{Coker } \Lambda g(m, n) \right) &= H_* \left( K\langle X(f) \rangle \otimes_S \left( E \otimes_R \text{Coker } \Lambda g(m, n) \right) \right) \\
&= H_* \left( \text{Coker } \Lambda g(m, n) \otimes_S (K\langle X(f) \rangle \otimes_R E) \right) \\
&= \text{Tor}_5^S \left( \text{Coker } \Lambda g(m, n), S \otimes_R E \right) = H_* \left( \Lambda f, E \right).^2
\end{align*}

**Corollary 2.5.** Let R be noetherian, and f, S be as above. Then depth(I(f); R) = m - n + 1 if and only if depth(I(X(f)); S/I(g(m, n))) = mn. If R is a local ring and every entry a_{ij} of the matrix of f is contained in the maximal ideal, m, of R (i.e., \( f(R^m) \subseteq mR^n \)), then depth (I(f); R) = m - n + 1 if and only if \( \{ X_{11} - a_{11}, \ldots, X_{mn} - a_{mn} \} \) is an \((S/I(g(m, n)))^m\)-sequence, where \( \bar{m} = (m, X_{11}, \ldots, X_{mn}) \).

**Proof.** The first statement follows from the above, namely \( H_* (\Lambda^m f, R) = H_* (X(f), S/I(g(m, n))) \) (letting \( p = n \)).

Now assume that R is local and all the a_{ij} are in m. Then

\[ H_* (X(f), S/I(g(m, n))) = 0 \]

if and only if \( H_i (X(f), S/I(g(m, n))) = 0 \) since \( \bar{m} \) is the unique maximal ideal of S containing I(X(f)). Since \( \{ X_{11} - a_{11}, \ldots, X_{mn} - a_{mn} \} \) is contained in \( \bar{m} \), \( H_* (X(f), S/I(g(m, n))) = 0 \) for all positive dimensions if and only if \( X_{11} - a_{11}, \ldots, X_{mn} - a_{mn} \) is an \((S/I(g(m, n)))^m\)-sequence.

Let R be a local ring, M an R-module of finite length, and \( R^m \to R^n \to M \to 0 \) an exact sequence. We have shown in [3, 3.4] that \( P_f (\nu, R) \) is an S-module such that \( \text{Tor}_i^S (A, S_f) \) is a polynomial function whose degree is \( m - 1 \)

---

^ The next to the last equality is obtained by using the following well-known fact: if S is a ring, \( Y = \{ Y_n \} \) an acyclic complex over the S-module B, and A an S-module such that \( \text{Tor}_i^S (A, Y_n) = 0 \) for all \( i > 0 \) and all \( n \), then \( \text{Tor}_i^S (A, B) = H_i (A \otimes_S Y) \).
+dim \, R, \text{ and that } e_R(M) = (n - 1 + \dim R)! \text{ (leading coefficient of } P_f(n, R)) \text{ is a positive integer which depends only on the module } M. \text{ Furthermore, we have shown } [3, 4.3] \text{ that }

\chi H_* \left( \psi, f, R \right) = \binom{n - 1}{n - \phi} e_R(M)

\text{if } f \text{ is a parameter matrix, i.e., if } m - n + 1 = \dim R, \text{ where } \chi H_* \text{ stands for the Euler-Poincaré characteristic. We observe that } f \text{ is a parameter matrix if and only if } I(X(f)) \text{ is an ideal of definition for } \tilde{S} \text{ generated by a system of parameters, where } S = R[X_{11}, \cdots, X_{mn}] \text{ and } \tilde{S} = (S/I(g(m, n)))^m. \text{ Therefore, Theorem 2.4 yields immediately}

\text{Corollary 2.6. Let } f: R^m \rightarrow R^n \text{ be a parameter matrix. Then } e_R(M) = e_S(\tilde{S}/I(X(f))), \text{ where } M = \text{Coker } f.

\textbf{Bibliography}


\textbf{Brandeis University}

\textbf{ERRATA, VOLUME 16}

Culbreth Sudler, Jr., \textit{A direct proof of two theorems on two-line partitions}, pp. 161–168.

Page 165, (3.2). For } a, \text{ read } \alpha_a.

Page 167, line -1. For 9 5 3 1, read 11 5 3 1.