A GENERALIZED KOSZUL COMPLEX. III. A REMARK ON GENERIC ACYCLICITY

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1. **Introduction.** Let R be a commutative ring, and $S = R[X_{11}, \cdots, X_{mn}]$ be the polynomial ring in mn indeterminates over R. If we denote by $g(m, n): S^m \to S^n$ the map whose matrix is (X_{ij}), we may consider the (generalized) Koszul complexes $K(\wedge^p g(m, n))$ for $1 \leqq p \leqq n$ [2], [3]. One of the first things we shall show is that these complexes are acyclic, i.e. $H_i(K(\wedge^p g(m, n))) = 0$ for $i > 0$. We will do this by using a result in a recent paper by Northcott [4], which says that if R is noetherian, then the ideal, I, generated by the subdeterminants of order n of the matrix (X_{ij}) is perfect, and depth$(I; S) = m - n + 1$.

Because the complexes $K(\wedge^p g(m, n))$ are acyclic, we say that the generalized Koszul complex is *generically acyclic*. This generic acyclicity enables us to interpret the homology and cohomology groups of the generalized Koszul complex as Tor and Ext over suitable rings, and also expresses these groups as the homology (and cohomology) of the classical Koszul complex. As a result, we are able to give easy proofs of the rigidity of these complexes [1], and also prove an assertion [2] concerning the annihilator of their homology groups.

Throughout this note, all rings are commutative and have an identity element.

2. **Generic acyclicity.** As in §1, we let $S = R[X_{11}, \cdots, X_{mn}]$, where X_{ij} are indeterminates over R, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ $(m \geqq n)$, and $g(m, n): S^m \to S^n$ is the map whose matrix is (X_{ij}). The ideal $I(m, n)$ (or I) generated by the subdeterminants of (X_{ij}) of order n is simply the image of the map $\wedge^n g(m, n): \wedge^n S^m \to \wedge^n S^n$.

THEOREM 2.1. *For each p $(1 \leqq p \leqq n)$, the complex $K(\wedge^p g(m, n))$* [3, §1] *has trivial homology in positive dimensions. Thus $K(\wedge^p g(m, n))$ is a free resolution of* Coker $\wedge^p g(m, n)$.

PROOF. We know, by Northcott's result quoted in §1, that, in case R is noetherian, we have depth$(I; S) = m - n + 1$ [3, §2]. Thus, if R is noetherian, the theorem follows immediately from Corollary 2.6 of [3]. In case R is not noetherian, we observe that $R = \lim_{\to} R_\alpha$ where R_α is noetherian, and hence $S = \lim_{\to} S_\alpha$, where $S_\alpha = R_\alpha[X_{11}, \cdots, X_{mn}]$.

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Therefore, $K(\wedge^p (g(m, n), S) = \lim_* K(\wedge^p g(m, n), S_\alpha)$ and we see that $K(\wedge^p g(m, n), S)$ is acyclic.

COROLLARY 2.2. *Given a map $f: R^m \to R^n$ with matrix (a_{ij}) and a finitely generated R-module E, we have*

$$H_*\left(\overset{p}{\wedge} f, E\right) = \mathrm{Tor}^S\left(\mathrm{Coker} \overset{p}{\wedge} g(m, n), \bar{S} \otimes_R E\right),$$

$$H^*\left(\overset{p}{\wedge} f, E\right) = \mathrm{Ext}_S\left(\mathrm{Coker} \overset{p}{\wedge} g(m, n), \mathrm{Hom}_R(\bar{S}, E)\right),$$

where $S = R[X_{11}, \cdots, X_{mn}]$ and $\bar{S} = S/I(X(f))$, with $I(X(f)) = (X_{11} - a_{11}, \cdots, X_{mn} - a_{mn})$. (For definitions of H_* and H^*, see [**3**, §1].)

PROOF. This follows immediately from the fact that $K(\wedge^p g(m, n))$ is a free resolution of Coker $\wedge^p g(m, n)$ and that $K(\wedge^p f) \otimes_R E = (K(\wedge^p g(m, n)) \otimes_S \bar{S}) \otimes_R E = K(\wedge^p g(m, n)) \otimes_S (\bar{S} \otimes_R E)$ and

$$\begin{aligned} \mathrm{Hom}_R\left(K\left(\overset{p}{\wedge} f\right), E\right) &= \mathrm{Hom}_R\left(K\left(\overset{p}{\wedge} g(m, n)\right) \otimes_S \bar{S}, E\right), \\ &= \mathrm{Hom}_S\left(K\left(\overset{p}{\wedge} g(m, n)\right), \mathrm{Hom}_R(\bar{S}, E)\right). \end{aligned}$$

COROLLARY 2.3. *$H(\wedge^p f, \cdot)$ is annihilated by* Ann Coker $\wedge^p f$ (*cf.* [**2**]).

As another application we obtain the rigidity of the complex $K(\wedge^p f)$.

THEOREM 2.4. *Let f and S be as in 2.2, with R noetherian. If $X(f)$: $S^{mn} \to S$ is the map whose $mn \times 1$ matrix is $(X_{11} - a_{11}, \cdots, X_{mn} - a_{mn})$, then $H_*(\wedge^p f, E) = H_*(X(f), E \otimes_R$ Coker $\wedge^p g(m, n))$. In particular, $H_s(\wedge^p f, E) = 0$ implies $H_t(\wedge^p f, E) = 0$ for all $t \geqq s$.*

PROOF. $K(\wedge^p g(m, n))$ is an acyclic resolution of Coker $\wedge^p g(m, n)$, regardless of the coefficient ring R. In particular, this is so for $R/\mathfrak{a}$ for any ideal $\mathfrak{a}$ in R, i.e., $K(\wedge^p g(m, n)) \otimes_R R/\mathfrak{a}$ is acyclic for any ideal $\mathfrak{a}$. Thus $\mathrm{Tor}^R(\mathrm{Coker} \wedge^p g(m, n), R/\mathfrak{a}) = 0$ for all ideals $\mathfrak{a}$, since $K(\wedge^p g(m, n))$ is S-free and, hence, an R-free resolution of Coker $\wedge^p g(m, n)$. Therefore Coker $\wedge^p g(m, n)$ is flat as an R-module.

Now the standard Koszul complex $K(X(f))$ is an S-free resolution of Coker $X(f) = \bar{S}$, since $I(X(f))$ is generated by an S-sequence (as can be seen by linear substitution). Since $\bar{S} = R$ as an R-module,

$K(X(f))\otimes_R E$ is an acyclic resolution of $\bar{S}\otimes_R E$, and each term of $K(X(f))\otimes_R E$ is a direct sum of copies of $S\otimes_R E$. Now

$$\operatorname{Tor}_i^S\left(\operatorname{Coker}\bigwedge^p g(m,n), S\otimes_R E\right)=\operatorname{Tor}_i^R\left(\operatorname{Coker}\bigwedge^p g(m,n), E\right)=0,$$

for all $i>0$, since Coker $\bigwedge^p g(m, n)$ is R-flat. Therefore,

$$\begin{aligned} &H_*\left(X(f), E\otimes_R \operatorname{Coker}\bigwedge^p g(m,n)\right)\\ &\quad = H_*\left(K(X(f))\otimes_S\left(E\otimes_R \operatorname{Coker}\bigwedge^p g(m,n)\right)\right)\\ &\quad = H_*\left(\operatorname{Coker}\bigwedge^p g(m,n)\otimes_S (K(X(f))\otimes_R E)\right)\\ &\quad = \operatorname{Tor}^S\left(\operatorname{Coker}\bigwedge^p g(m,n), \bar{S}\otimes_R E\right) = H_*\left(\bigwedge^p f, E\right).^2 \end{aligned}$$

COROLLARY 2.5. *Let R be noetherian, and f, S be as above. Then* $\operatorname{depth}(I(f); R)=m-n+1$ *if and only if* $\operatorname{depth}(I(X(f)); S/I(g(m,n)))=mn$. *If R is a local ring and every entry a_{ij} of the matrix of f is contained in the maximal ideal,* $\mathfrak{m}$, *of R (i.e., $f(R^m)\subset \mathfrak{m}R^n$), then* depth $(I(f); R)=m-n+1$ *if and only if* $\{X_{11}-a_{11},\cdots,X_{mn}-a_{mn}\}$ *is an $(S/I(g(m,n)))_{\tilde{\mathfrak{m}}}$-sequence, where $\tilde{\mathfrak{m}}=(\mathfrak{m}, X_{11},\cdots,X_{mn})$.*

PROOF. The first statement follows from the above, namely $H_*(\bigwedge^n f, R)=H_*(X(f), S/I(g(m,n)))$ (letting $p=n$).

Now assume that R is local and all the a_{ij} are in $\mathfrak{m}$. Then

$$H_i(X(f), S/I(g(m,n))) = 0$$

if and only if $H_i(X(f), S/I(g(m,n)))_{\tilde{\mathfrak{m}}}=0$ since $\tilde{\mathfrak{m}}$ is the unique maximal ideal of S containing $I(X(f))$. Since $(X_{11}-a_{11},\cdots,X_{mn}-a_{mn})$ is contained in $\tilde{\mathfrak{m}}$, $H_*(X(f), S/I(g(m,n)))=0$ for all positive dimensions if and only if $X_{11}-a_{11},\cdots,X_{mn}-a_{mn}$ is an $(S/I(g(m,n)))_{\tilde{\mathfrak{m}}}$-sequence.

Let R be a local ring, M an R-module of finite length, and $R^m\xrightarrow{f}R^n\to M\to 0$ an exact sequence. We have shown in [3, 3.4] that $P_f(\nu, R)=l(\operatorname{Coker} S_\nu(f))$ is a polynomial function whose degree is $n-1$

[2] The next to the last equality is obtained by using the following well-known fact: if S is a ring, $Y=\{Y_n\}$ an acyclic complex over the S-module B, and A an S-module such that $\operatorname{Tor}_i{}^S(A, Y_n)=0$ for all $i>0$ and all n, then $\operatorname{Tor}_i{}^S(A,B)=H_i(A\otimes_S Y)$.

$+\dim R$, and that $e_R(M)=(n-1+\dim R)!$ (leading coefficient of $P_f(\nu, R)$) is a positive integer which depends only on the module M. Furthermore, we have shown [3, 4.3] that

$$\chi H_*\left(\overset{p}{\wedge} f, R\right) = \binom{n-1}{n-p} e_R(M)$$

if f is a parameter matrix, i.e., if $m-n+1=\dim R$, where χH_* stands for the Euler-Poincaré characteristic. We observe that f is a parameter matrix if and only if $I(X(f))$ is an ideal of definition for $\tilde{S}$ generated by a system of parameters, where $S=R[X_{11}, \cdots, X_{mn}]$ and $\tilde{S}=(S/I(g(m, n)))_{\tilde{\mathfrak{m}}}$. Therefore, Theorem 2.4 yields immediately

COROLLARY 2.6. *Let $f\colon R^m \to R^n$ be a parameter matrix. Then $e_R(M) = e_{\tilde{S}}(\tilde{S}/I(X(f)))$, where $M=$* Coker *f.*

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Culbreth Sudler, Jr., *A direct proof of two theorems on two-line partitions*, pp. 161–168.

Page 165, (3.2). For a_ν read α_ν.

Page 167, line -1. For 9 5 3 1, read 11 5 3 1.