

REFERENCES

1. H.-J. Hoehnke, *Zur Theorie der Gruppoiden*. I, Math. Nachr. **24** (1962), 137–168; III, Acta Math. **13** (1962), 91–100.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I. 1961.

TULANE UNIVERSITY

**EVERY STANDARD CONSTRUCTION IS INDUCED
BY A PAIR OF ADJOINT FUNCTORS**

H. KLEISLI

In this note, we prove the converse of the following result of P. Huber [2]. Let $F: \mathcal{K} \rightarrow \mathcal{L}$ and $G: \mathcal{L} \rightarrow \mathcal{K}$ be covariant *adjoint functors*, that is, functors such that there exist two (functor) morphisms $\zeta: I \rightarrow GF$ and $\eta: FG \rightarrow I$ satisfying the relations

- (1) $(\eta * F) \circ (F * \zeta) = \iota * F,$
- (2) $(G * \eta) \circ (\zeta * G) = \iota * G.$

Then, the triple (C, k, p) given by

$$C = FG, \quad k = \eta \quad \text{and} \quad p = F * \zeta * G$$

is a *standard construction* in \mathcal{L} , that is, C is a covariant functor, $k: C \rightarrow I$ and $p: C \rightarrow C^2$ are (functor) morphisms, and the following relations hold:

- (3) $(k * C) \circ p = (C * k) \circ p = \iota * C,$
- (4) $(p * C) \circ p = (C * p) \circ p.$

This standard construction is said to be *induced by the pair of adjoint functors F and G* . For further explanation of the notation and terminology, see [2], or the appendix of [1].

THEOREM. *Let (C, k, p) be a standard construction in a category \mathcal{L} . Then there exists a category \mathcal{K} and two covariant functors $F: \mathcal{K} \rightarrow \mathcal{L}$ and $G: \mathcal{L} \rightarrow \mathcal{K}$ such that*

- (i) F is (left) adjoint to G ,
- (ii) (C, k, p) is induced by F and G .

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The category \mathcal{K} is given as follows. The objects of \mathcal{K} are the same as those of \mathcal{L} . For each pair A, A' of objects, we define

$$\text{Hom}_{\mathcal{K}}(A, A') = \text{Hom}_{\mathcal{L}}(CA, A').$$

For each triple A, A', A'' of objects, and each pair of morphisms $\alpha \in \text{Hom}_{\mathcal{K}}(A, A')$ and $\alpha' \in \text{Hom}_{\mathcal{K}}(A', A'')$, the composition, $\alpha' \cdot \alpha \in \text{Hom}_{\mathcal{K}}(A, A'')$ is given by

$$\alpha' \cdot \alpha = \alpha' \circ C\alpha \circ pA.$$

The identity $\iota_A \in \text{Hom}_{\mathcal{K}}(A, A)$ is defined by setting

$$\iota_A = kA: CA \rightarrow A.$$

The associativity and identity laws follow from (4) and (3). By (4), we have

$$\begin{aligned} \alpha'' \cdot (\alpha' \cdot \alpha) &= \alpha'' \circ C(\alpha' \circ C\alpha \circ pA) \circ pA \\ &= \alpha'' \circ C\alpha' \circ C^2\alpha \circ ((C * p) \circ p)A \\ &= \alpha'' \circ C\alpha' \circ C^2\alpha \circ ((p * C) \circ p)A \\ &= (\alpha'' \circ C\alpha' \circ pA') \circ C\alpha \circ pA = (\alpha'' \cdot \alpha') \cdot \alpha. \end{aligned}$$

By (3), $\alpha \cdot \iota_A = \alpha \circ CkA \circ pA = \alpha \circ ((C * k) \circ p)A = \alpha \circ (\iota * C)A = \alpha$, and, similarly, $\iota_A \cdot \alpha = \alpha$.

The functor C can be factored as follows:

$$\begin{array}{ccc} & C & \\ \mathcal{L} & \longrightarrow & \mathcal{L}, \\ & G \searrow & \nearrow F \\ & \mathcal{K} & \end{array}$$

where G and F are covariant functors given by $GA = A$ and $G\alpha = \alpha \circ kA$ for every object A and morphism α of \mathcal{L} , $FB = CB$ and $F\beta = C\beta \circ pB$ for every object B and morphism β of \mathcal{K} . The functor properties of G and F are immediate consequences of (3), (4) and of the definition of the identities in \mathcal{K} . The verifications are straightforward.

In order to show that F is (left) adjoint to G , and that (C, k, p) is induced by F and G , put $\eta = k$ and define $\zeta B = \iota_{CB}: CB \rightarrow CB$ for every object B of \mathcal{K} . The family $(\zeta B)_{B \in \mathcal{K}}$ yields a (functor) morphism $\zeta: I \rightarrow GF$. Indeed, let $\beta \in \text{Hom}_{\mathcal{K}}(B, B')$; then, $\zeta B' \cdot \beta = \iota_{CB'} \circ C\beta \circ pB = C\beta \circ pB = C\beta \circ C\iota_{CB} \circ pB = GF\beta \cdot \zeta B$. Clearly, $C = FG$, and by definition $k = \eta$. Moreover, for each object A of \mathcal{L} ,

$$(F * \zeta * G)A = F\iota_{CGA} = C\iota_{CA} \circ pA = pA;$$

hence $p = F * \zeta * G$. Using (3), we obtain

$$\begin{aligned} ((\eta * F) \circ (F * \zeta)) * G &= (\eta * FG) \circ (F * \zeta * G) = (k * C) \circ p \\ &= \iota * C = (\iota * F) * G. \end{aligned}$$

It is easily seen that the factor G may be cancelled. Thus, relation (1) holds. Furthermore, we have

$$\begin{aligned} F * ((G * \eta) \cdot (\zeta * G)) &= (FG * \eta) \circ (F * \zeta * G) = (C * k) \circ p \\ &= \iota * C = F * (\iota * G). \end{aligned}$$

Here, the factor F may be cancelled. Indeed, let β_1 and β_2 be elements of $\text{Hom}_{\mathcal{G}}(B, B')$ such that $F\beta_1 = F\beta_2$. By (3),

$$\begin{aligned} kB' \circ F\beta_1 &= kB' \circ C\beta_1 \circ pB = \beta_1 \circ kCB \circ pB = \beta_1 \circ ((k * C) \circ p)B \\ &= \beta_1 \circ (\iota * C)B = \beta_1, \end{aligned}$$

and, similarly, $kB' \circ F\beta_2 = \beta_2$. Hence $\beta_1 = \beta_2$. Therefore, relation (2) holds.

REFERENCES

1. R. Godement, *Théorie des faisceaux*, Actualités Sci. Indust. No. 1252, Hermann, Paris, 1958.
2. P. Huber, *Homotopy theory in general categories*, Math. Ann. 144 (1961), 361–385.

UNIVERSITY OF OTTAWA