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TULANE UNIVERSITY

# EVERY STANDARD CONSTRUCTION IS INDUCED BY A PAIR OF ADJOINT FUNCTORS

H. KLEISLI

In this note, we prove the converse of the following result of P. Huber [2]. Let $F\colon \mathfrak{K}\to\mathfrak{L}$ and $G\colon \mathfrak{L}\to\mathfrak{K}$ be covariant *adjoint functors*, that is, functors such that there exist two (functor) morphisms $\zeta\colon I\to GF$ and $\eta\colon FG\to I$ satisfying the relations

$$(\eta * F)\circ(F*\zeta) = \iota * F, \tag{1}$$

$$(G*\eta)\circ(\zeta * G) = \iota * G. \tag{2}$$

Then, the triple $(C, k, p)$ given by

$$C = FG, \qquad k = \eta \quad\text{and}\quad p = F*\zeta*G$$

is a *standard construction* in $\mathfrak{L}$, that is, $C$ is a covariant functor, $k\colon C\to I$ and $p\colon C\to C^2$ are (functor) morphisms, and the following relations hold:

$$(k*C)\circ p = (C*k)\circ p = \iota * C, \tag{3}$$

$$(p*C)\circ p = (C*p)\circ p. \tag{4}$$

This standard construction is said to be *induced by the pair of adjoint functors F and G.* For further explanation of the notation and terminology, see [2], or the appendix of [1].

THEOREM. *Let $(C, k, p)$ be a standard construction in a category $\mathfrak{L}$. Then there exists a category $\mathfrak{K}$ and two covariant functors $F\colon \mathfrak{K}\to\mathfrak{L}$ and $G\colon \mathfrak{L}\to\mathfrak{K}$ such that*

(i) *$F$ is (left) adjoint to $G$,*

(ii) *$(C, k, p)$ is induced by $F$ and $G$.*

Received by the editors March 2, 1964.

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The category $\mathcal{K}$ is given as follows. The objects of $\mathcal{K}$ are the same as those of $\mathcal{L}$. For each pair $A$, $A'$ of objects, we define

$$\mathrm{Hom}_{\mathcal{K}}(A, A') = \mathrm{Hom}_{\mathcal{L}}(CA, A').$$

For each triple $A$, $A'$, $A''$ of objects, and each pair of morphisms $\alpha \in \mathrm{Hom}_{\mathcal{K}}(A, A')$ and $\alpha' \in \mathrm{Hom}_{\mathcal{K}}(A', A'')$, the composition, $\alpha' \cdot \alpha \in \mathrm{Hom}_{\mathcal{K}}(A, A'')$ is given by

$$\alpha' \cdot \alpha = \alpha' \circ C\alpha \circ pA.$$

The identity $\mathbf{\iota}_A \in \mathrm{Hom}_{\mathcal{K}}(A, A)$ is defined by setting

$$\mathbf{\iota}_A = kA : CA \to A.$$

The associativity and identity laws follow from (4) and (3). By (4), we have

$$\begin{aligned}
\alpha'' \cdot (\alpha' \cdot \alpha) &= \alpha'' \circ C(\alpha' \circ C\alpha \circ pA) \circ pA \\
&= \alpha'' \circ C\alpha' \circ C^2\alpha \circ ((C * p) \circ p)A \\
&= \alpha'' \circ C\alpha' \circ C^2\alpha \circ ((p * C) \circ p)A \\
&= (\alpha'' \circ C\alpha' \circ pA') \circ C\alpha \circ pA = (\alpha'' \cdot \alpha') \cdot \alpha.
\end{aligned}$$

By (3), $\alpha \cdot \mathbf{\iota}_A = \alpha \circ CkA \circ pA = \alpha \circ ((C * k) \circ p)A = \alpha \circ (\iota * C)A = \alpha$, and, similarly, $\mathbf{\iota}_A \cdot \alpha = \alpha$.

The functor $C$ can be factored as follows:

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "C"] \arrow[dr, "G"'] & & \mathcal{L}, \\
& \mathcal{K} \arrow[ur, "F"'] &
\end{tikzcd}$$

where $G$ and $F$ are covariant functors given by $GA = A$ and $G\alpha = \alpha \circ kA$ for every object $A$ and morphism $\alpha$ of $\mathcal{L}$, $FB = CB$ and $F\beta = C\beta \circ pB$ for every object $B$ and morphism $\beta$ of $\mathcal{K}$. The functor properties of $G$ and $F$ are immediate consequences of (3), (4) and of the definition of the identities in $\mathcal{K}$. The verifications are straightforward.

In order to show that $F$ is (left) adjoint to $G$, and that $(C, k, p)$ is induced by $F$ and $G$, put $\eta = k$ and define $\zeta B = \iota_{CB}: CB \to CB$ for every object $B$ of $\mathcal{K}$. The family $(\zeta B)_{B \in \mathcal{K}}$ yields a (functor) morphism $\zeta: I \to GF$. Indeed, let $\beta \in \mathrm{Hom}_{\mathcal{K}}(B, B')$; then, $\zeta B' \cdot \beta = \iota_{CB'} \circ C\beta \circ pB = C\beta \circ pB = C\beta \circ C\iota_{CB} \circ pB = GF\beta \cdot \zeta B$. Clearly, $C = FG$, and by definition $k = \eta$. Moreover, for each object $A$ of $\mathcal{L}$,

$$(F * \zeta * G)A = F\iota_{CGA} = C\iota_{CA} \circ pA = pA;$$

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hence $p = F * \zeta * G$. Using (3), we obtain

$$((\eta * F) \circ (F * \zeta)) * G = (\eta * FG) \circ (F * \zeta * G) = (k * C) \circ p$$
$$= \iota * C = (\iota * F) * G.$$

It is easily seen that the factor $G$ may be cancelled. Thus, relation (1) holds. Furthermore, we have

$$F * ((G * \eta) \cdot (\zeta * G)) = (FG * \eta) \circ (F * \zeta * G) = (C * k) \circ p$$
$$= \iota * C = F * (\iota * G).$$

Here, the factor $F$ may be cancelled. Indeed, let $\beta_1$ and $\beta_2$ be elements of $\mathrm{Hom}_{\mathcal{K}}(B, B')$ such that $F\beta_1 = F\beta_2$. By (3),

$$kB' \circ F\beta_1 = kB' \circ C\beta_1 \circ pB = \beta_1 \circ kCB \circ pB = \beta_1 \circ ((k * C) \circ p)B$$
$$= \beta_1 \circ (\iota * C)B = \beta_1,$$

and, similarly, $kB' \circ F\beta_2 = \beta_2$. Hence $\beta_1 = \beta_2$. Therefore, relation (2) holds.

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University of Ottawa

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