PARTIAL HOMOMORPHIC IMAGES OF BRANDT GROUPOIDS[1]

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The main purpose of the present note is to show (Theorem 2) that any regular $\mathfrak{D}$-class of any semigroup is a partial homomorphic image of a Brandt groupoid. It follows from this that a semigroup with zero is a partial homomorphic image of a Brandt semigroup if and only if it is regular and 0-bisimple.

In the first section, an alternative formulation is given of the determination by H.-J. Hoehnke [1] of all partial homomorphisms of a Brandt groupoid into an arbitrary semigroup. This is first done (Theorem 1) for any completely 0-simple semigroup. The result is a straightforward generalization of Theorem 3.14 of [2], in which all partial homomorphisms of one completely 0-simple semigroup into another are determined. The present terminology is that of [2]; Hoehnke omits the adjective "partial." Basic definitions given in [2] will not be repeated here; likewise, references to the fundamental work of Brandt, Rees, Green, and Munn can be found in [2].

1. Partial homomorphisms of a completely 0-simple semigroup. Let S and S^* be semigroups with zero elements 0 and 0^*, respectively. A mapping θ of S into S^* is called a *partial homomorphism* if (i) $0\theta = 0^*$, and (ii) $(ab)\theta = (a\theta)(b\theta)$ for every pair of elements a, b of S such that $ab \neq 0$. The restriction of θ to $S\backslash 0$ is then a partial homomorphism of the partial groupoid $S\backslash 0$ into S^* as defined in [2, p. 93]. By agreeing to the trivial convention (i), there is no essential distinction between partial homomorphisms of S into S^* and of $S\backslash 0$ into S^*. Moreover, we need not require that S^* have a zero element; if it does not, we adjoin a zero element 0^* to it for the application of (i).

The author's interest in partial homomorphisms originated in the fact that they arise naturally in the theory of extensions of semigroups [2, §4.4].

A partial homomorphism $\theta: S \rightarrow S^*$ evidently preserves regularity [2, p. 26] and Green's relations $\mathfrak{R}$, $\mathfrak{L}$, $\mathfrak{D}$, and $\mathfrak{H}$ [2, p. 47]. It follows that if S is regular and 0-bisimple (i.e., $S\backslash 0$ is a $\mathfrak{D}$-class of S [2, p. 76]), then $(S\backslash 0)\theta$ is contained in a regular $\mathfrak{D}$-class D of S^*. This is the case, in particular, if S is completely 0-simple [2, Theorem

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2.51, p. 79]. Since a Brandt semigroup B^0 is just a completely 0-simple inverse semigroup [2, Theorem 3.9, p. 102], we conclude, finally, that if θ is a partial homomorphism of a Brandt groupoid $B=B^0\backslash 0$ into a semigroup S^*, then $B\theta$ is contained in a regular $\mathfrak{D}$-class D of S. One might think that these successive particularizations would result in some restriction on D, particularly if θ is onto; the object of this note is to show that this is not the case (Theorem 2 below).

Let D be a regular $\mathfrak{D}$-class of S^*. Let

$$\{R_{i^*}: i^* \in I^*\} \quad \text{and} \quad \{L_{\lambda^*}: \lambda^* \in \Lambda^*\}$$

be the $\mathfrak{R}$-classes and $\mathfrak{L}$-classes, respectively, of S^* contained in D. Then $H_{i^*\lambda^*}=R_{i^*}\cap L_{\lambda^*}$ are the $\mathfrak{H}$-classes of S^* contained in D. We know that at least one of these must contain an idempotent, and so be a maximal subgroup of S^* [2, Theorem 2.16, p. 59]; choose one such and call it $H^*=H_{1^*1^*}$, 1^* being an element of $I^*\cap\Lambda^*$. For each i^* in I^*, pick r_{i^*} in $H_{i^*1^*}$, and for each λ^* in Λ^* pick q_{λ^*} in $H_{1^*\lambda^*}$. Then [2, Theorem 3.4, p. 92], every element of D is uniquely representable in the form

$$(1) \qquad r_{i^*}xq_{\lambda^*} \qquad (x \in H^*;\ i^* \in I^*,\ \lambda^* \in \Lambda^*).$$

We regard the triple $(x; i^*, \lambda^*)$ as coordinates of the element (1).

By the Rees Theorem [2, Theorem 3.5, p. 94], a completely 0-simple semigroup can be represented as a regular Rees $I\times\Lambda$ matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$ over a group with zero G^0, and with $\Lambda\times I$ sandwich matrix $P=(p_{\lambda i})$. The elements of $\mathfrak{M}^0$ can be represented as triples $(a; i, \lambda)$ multiplying according to

$$(2) \quad (a; i, \lambda)(b; j, \mu) = (ap_{\lambda j}b; i, \mu) \qquad (a, b \in G^0; i, j \in I;\ \lambda, \mu \in \Lambda).$$

In fact, the proof of the Rees Theorem amounts to coordinatizing the $\mathfrak{D}$-class $\mathfrak{M}^0\backslash 0$. It should be remarked that, for an arbitrary regular $\mathfrak{D}$-class D, the elements (1) do not have a simple law of multiplication like (2).

THEOREM 1. *Let S be a completely 0-simple semigroup represented as a regular Rees $I\times\Lambda$ matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$. Let θ be a partial homomorphism of S into a semigroup S^*. Then $(S\backslash 0)\theta$ is contained in a regular $\mathfrak{D}$-class D of S. Let D be coordinatized as in* (1). *Then*

$$(3) \qquad (a; i, \lambda)\theta = r_{i\phi}u_i(a\omega)v_\lambda q_{\lambda\psi} \qquad (a \in G; i \in I, \lambda \in \Lambda),$$

where (i) $\phi: I\rightarrow I^*$ *and* $\psi: \Lambda\rightarrow\Lambda^*$ *are mappings such that if* $p_{\lambda i}\neq 0$ *then* $q_{\lambda\psi}r_{i\phi}\in H^*$;

(ii) $\omega: G\rightarrow H^*$ *is a* (*group*) *homomorphism*;

(iii) *$u: I \rightarrow H^*$ and $v: \Lambda \rightarrow H^*$ are mappings such that if $p_{\lambda i} \neq 0$ then*

$$p_{\lambda i}\omega = v_\lambda(q_{\lambda\psi} r_{i\phi}) u_i. \tag{4}$$

The mappings ϕ, ψ, ω, u, v are uniquely determined by θ. Conversely, if mappings ϕ, ψ, ω, u, v are given satisfying (i), (ii), *and* (iii), *then* (3) *defines a partial homomorphism θ of $S\backslash 0$ into D.*

Proof. The proof is so much like that of Theorem 3.14 of [2, p. 109], that we give only the outline. We can assume that the entry p_{11} of P is not zero. The mappings ϕ and ψ are determined by

$$R_i\theta \subseteq R_{i\phi}, \qquad L_\lambda\theta \subseteq L_{\lambda\psi},$$

where $\{R_i: i \in I\}$ are the $\mathcal{R}$-classes, and $\{L_\lambda: \lambda \in \Lambda\}$ are the $\mathcal{L}$-classes, of S. This implies that

$$(a; i, \lambda)\theta = r_{i\phi} x q_{\lambda\psi}$$

for some x in H^*. If $p_{\lambda i} \neq 0$, then $(p_{\lambda i}^{-1}; i, \lambda)\theta$ is an idempotent in $H_{i\phi,\lambda\psi}$, and it follows that $q_{\lambda\psi} r_{i\phi} \in H^*$ [2, Theorem 2.17, p. 59]. Defining $\omega: G \rightarrow H^*$ by

$$(p_{11}^{-1}, a; 1, 1)\theta = r_{1\phi} h_0^{-1}(a\omega) q_{1\psi} \qquad (h_0 = q_{1\psi} r_{1\phi}),$$

a brief calculation, using the uniqueness of the representation (1), shows that ω is a homomorphism. For each $i \in I$ and $\lambda \in \Lambda$ we define u_i and v_λ in H^* by

$$(e; i, 1)\theta = r_{i\phi} u_i q_{1\psi},$$
$$(p_{11}^{-1} 1, \lambda)\theta = r_{1\phi} h_0^{-1} v_\lambda q_{\lambda\psi}.$$

Applying θ to

$$(a; i, \lambda) = (e; i, 1)(p_{11}^{-1} a; 1, 1)(p_{11}^{-1}; 1, \lambda),$$

we obtain (3). Applying θ to (2) and using (3), again with the uniqueness of (1), we obtain (4). This last step can be inverted to yield the converse part of the theorem.

From a constructive point of view, Theorem 1 has the drawback that, for given ϕ, ψ, and ω satisfying (i) and (ii), there is no assurance that u and v can be found so as to satisfy (iii). This drawback disappears, however, when S is a Brandt semigroup B^0. Here we can assume $B^0 = \mathfrak{M}^0(G; I, I; \Delta)$, where $\Delta = (\delta_{ij})$ is the $I \times I$ identity matrix over G^0 [2, Theorem 3.9, p. 102]. The condition (4) now reduces to

$$e^* = v_i q_{i\psi} r_{i\phi} u_i \qquad (\text{all } i \in I),$$

where e^* is the identity element of H^*; or, what is equivalent, to

$$v_i = u_i^{-1}(q_{i\psi}r_{i\phi})^{-1}. \tag{5}$$

We note that $q_{i\psi}r_{i\phi} \in H^*$ by (i). Thus we can always satisfy (iii) by choosing $u: I \to H^*$ arbitrarily, and then defining $v: I \to H^*$ by (5). Formula (3) becomes

$$(a; i, j)\theta = r_{i\phi}u_i(a\omega)u_j^{-1}(q_{j\psi}r_{j\phi})^{-1}q_{j\psi}. \tag{6}$$

This differs from Hoehnke's formula (16) of [1, Part III, p. 97], chiefly because a definite coordinate system has been chosen for D, independent of θ.

Now let D itself be a Brandt groupoid, say

$$D = B^* = \mathfrak{M}^0(H^*; I^*, I^*; \Delta^*)\backslash 0.$$

Let us use square brackets to represent the elements $[x^*; i^*, j^*]$ of B^*. It is natural to choose $r_{i^*} = [e^*; i^*, 1^*]$ and $q_{i^*} = [e^*; 1^*, i^*]$. We then have $q_{i^*}r_{i^*} = [e^*; 1^*, 1^*]$, while $q_{i^*}r_{j^*} = 0$ in B^{*0}, or is undefined in B^*, if $i^* \neq j^*$. Hence condition (i) of Theorem 1 requires that $i\psi = i\phi$ for every i in I; that is, $\psi = \phi$. (5) becomes simply $v_i = u_i^{-1}$, and (6) becomes

$$(a; i, j)\theta = [u_i(a\omega)u_j^{-1}; i\phi, j\phi]. \tag{7}$$

Thus every partial homomorphism θ of one Brandt groupoid, B, into another, B^*, is given by (7) in terms of (i) an arbitrary mapping $\phi: I \to I^*$, (ii) an arbitrary homomorphism $\omega: G \to H^*$, and (iii) an arbitrary mapping $u: I \to H^*$. (7) is equivalent to Hoehnke's formula (22) in [1, Part I, p. 164]. It can also be obtained by specialization from Theorem 3.14 of [2].

2. **Partial homomorphic images of Brandt groupoids.** We come now to the main result of the present note.

THEOREM 2. *Any regular $\mathfrak{D}$-class of any semigroup is a partial homomorphic image of some Brandt groupoid.*

PROOF. Let D be a regular $\mathfrak{D}$-class of a semigroup S. Let

$$\{R_i: i \in I\} \quad \text{and} \quad \{L_\lambda: \lambda \in \Lambda\}$$

be the $\mathfrak{R}$-classes and $\mathfrak{L}$-classes, respectively, of S contained in D. As usual, we may assume that I and Λ have an element 1 in common such that $H_{11} = R_1 \cap L_1$ is a group. But now we shall also assume, as we evidently may, that I and Λ are otherwise disjoint: $I \cap \Lambda = \{1\}$.

As usual, choose r_i in H_{i1} and q_λ in $H_{1\lambda}$ in any way, for i in $I\backslash 1$ and λ in $\Lambda\backslash 1$, and choose $r_1 = q_1 = e_{11}$, the identity element of H_{11}. As in (1), without the stars, every element of D is uniquely representable in the form

$$r_i a q_\lambda \qquad (a \in H_{11};\ i \in I,\ \lambda \in \Lambda). \tag{8}$$

For i in $I\backslash 1$ and λ in $\Lambda\backslash 1$, let q_i be any inverse of r_i in R_1, and let r_λ be any inverse of q_λ in L_1. Then

$$q_\alpha r_\alpha = e_{11} \qquad (\text{all } \alpha \text{ in } I \cup \Lambda). \tag{9}$$

Let $B = \mathfrak{M}^0(H_{11};\ I\cup\Lambda,\ I\cup\Lambda;\ \Delta)\backslash 0$. Denote the elements of B by triples $(a;\alpha,\beta)$. Multiplication in B is given by

$$(a;\alpha,\beta)(b;\beta,\gamma) = (ab;\alpha,\gamma) \qquad (a, b \in H_{11};\ \alpha,\beta,\gamma \in I \cup \Lambda). \tag{10}$$

Products $(a;\alpha,\beta)(b;\beta',\gamma)$ with $\beta\neq\beta'$ are not defined in B (and are zero in B^0). Define $\theta\colon B\to D$ as follows:

$$(a;\alpha,\beta)\theta = r_\alpha a q_\beta \qquad (a \in H_{11};\ \alpha,\beta \in I \cup \Lambda).$$

Then, because of (9),

$$\begin{aligned}(a;\alpha,\beta)\theta(b;\beta,\gamma)\theta &= r_\alpha a q_\beta r_\beta b q_\gamma = r_\alpha a b q_\gamma \\ &= (ab;\alpha,\gamma)\theta.\end{aligned}$$

From this and (10), it follows that θ is a partial homomorphism of B into D. Moreover, $B\theta = D$, since $B\theta$ contains all the elements $r_i a q_\lambda$ of (8).

As described in §3.3 of [2], if we adjoin a zero element 0 to a Brandt groupoid B, defining $ab=0$ if ab is undefined in B, we obtain a Brandt semigroup B^0, that is, a completely 0-simple inverse semigroup. The following is immediate from Theorem 2 and the first assertion in Theorem 1.

COROLLARY 1. *A semigroup with zero is a partial homomorphic image of some Brandt semigroup if and only if it is regular and* 0*-bisimple.*

As defined in [2, p. 93], a *partial isomorphism* is a partial homomorphism which is one-to-one and onto. Not every regular $\mathfrak{D}$-class is a partial isomorphic image of some Brandt groupoid, and the question of telling which ones are remains unsettled. The next theorem gives a sufficient condition.

THEOREM 3. *Let D be a regular $\mathfrak{D}$-class of a semigroup S with the property that it is possible to set up a one-to-one correspondence between the $\mathfrak{R}$-classes R of D and the $\mathfrak{L}$-classes $\mathfrak{L}$ of D such that if R and L correspond, then $R\cap L$ contains an idempotent. Then D is a partial isomorphic image of the Brandt groupoid having the same structure group as D and the same number of $\mathfrak{R}$-classes (and $\mathfrak{L}$-classes) as D.*

PROOF. By hypothesis, we can index the $\mathcal{R}$-classes and the $\mathcal{L}$-classes of D by the same index set I, such that for each i in I, $R_i \cap L_i$ contains an idempotent e_i. The $\mathcal{H}$-class $H_{ii} = R_i \cap L_i$ is then the maximal subgroup H_{e_i} of S containing e_i. Let $1 \in I$, and pick q_i in H_{1i} in any way for i in $I \backslash 1$, and let $q_1 = e_1$. Let q_i' be the inverse of q_i in H_{i1}; such exists since both H_{11} and H_{ii} contain idempotents [2, Theorem 2.18, p. 60]. Take $B = \mathfrak{M}^0(H_{11}; I, I; \Delta) \backslash 0$ and define $\theta: B \to D$ by

$$(a; i, j)\theta = q_i' a q_j \qquad (a \in H_{11}; i, j \in I). \tag{11}$$

Since every element of D is uniquely expressible in the form on the right-hand side of (11), and $q_j q_j' = e_1$, we see at once that θ is a partial isomorphism of B onto D.

B is unique, to within isomorphism, since any Brandt groupoid is completely determined by its structure group and the cardinal number of its $\mathcal{R}$-classes (or $\mathcal{L}$-classes).

COROLLARY 2. *Every 0-bisimple inverse semigroup S is a partial isomorphic image of the Brandt semigroup having the same structure group as S and the same number of idempotents as S.*

PROOF. The hypothesis of Theorem 3 is satisfied by any inverse semigroup [2, Corollary 2.19, p. 60]. For 0-bisimple inverse semigroups, in particular, for Brandt semigroups, the sets of $\mathcal{R}$-classes, $\mathcal{L}$-classes, and nonzero idempotents all have the same cardinal.

We conclude with an example to show that a regular 0-bisimple semigroup may be a partial isomorphic image of a Brandt semigroup, but not of one having the same structure group.

Let $B = \mathfrak{M}^0(E; I, I; \Delta) \backslash 0$, where $E = \{e\}$ is a one-element group, and $I = \{1, 2\}$. Let $S \backslash 0 = H \times E$, where H is a cyclic group $\{e, a\}$ of order 2, and E is a right zero semigroup of order 2. We may represent the elements of S as pairs $(x; i)$ with $x \in H$, $i \in I$, multiplying as follows:

$$(x; i)(y; j) = (xy; j) \qquad (x, y \in H; i, j \in I).$$

Define $\theta: B^0 \to S$ by

$$(e; 1, 1)\theta = (e; 1), \qquad (e; 1, 2)\theta = (a; 2)$$
$$(e; 2, 1)\theta = (a; 1), \qquad (e; 2, 2)\theta = (e; 2)$$

and $0\theta = 0$. Clearly θ is one-to-one and onto, and it is easy to verify that it is a partial homomorphism.

On the other hand, S cannot be a partial isomorphic image of any Brandt semigroup B^0 having structure group of order 2. For B must then have order twice a square, whereas $S \backslash 0$ has order 4.

References

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EVERY STANDARD CONSTRUCTION IS INDUCED BY A PAIR OF ADJOINT FUNCTORS

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In this note, we prove the converse of the following result of P. Huber [2]. Let $F\colon \mathcal{K}\to\mathcal{L}$ and $G\colon \mathcal{L}\to\mathcal{K}$ be covariant *adjoint functors*, that is, functors such that there exist two (functor) morphisms $\zeta\colon I\to GF$ and $\eta\colon FG\to I$ satisfying the relations

$$(\eta * F)\circ(F*\zeta) = \iota * F, \tag{1}$$

$$(G*\eta)\circ(\zeta * G) = \iota * G. \tag{2}$$

Then, the triple (C, k, p) given by

$$C = FG, \qquad k = \eta \quad \text{and} \quad p = F*\zeta*G$$

is a *standard construction* in $\mathcal{L}$, that is, C is a covariant functor, $k\colon C\to I$ and $p\colon C\to C^2$ are (functor) morphisms, and the following relations hold:

$$(k*C)\circ p = (C*k)\circ p = \iota * C, \tag{3}$$

$$(p*C)\circ p = (C*p)\circ p. \tag{4}$$

This standard construction is said to be *induced by the pair of adjoint functors F and G*. For further explanation of the notation and terminology, see [2], or the appendix of [1].

Theorem. *Let (C, k, p) be a standard construction in a category $\mathcal{L}$. Then there exists a category $\mathcal{K}$ and two covariant functors $F\colon \mathcal{K}\to\mathcal{L}$ and $G\colon \mathcal{L}\to\mathcal{K}$ such that*

(i) *F is (left) adjoint to G,*

(ii) *(C, k, p) is induced by F and G.*

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