SOME REMARKS ON SYMPLECTIC AUTOMORPHISMS

GEORGE W. MACKEY

1. Introduction. Let G be a separable[1] locally compact abelian group and let $\hat{G}$ be its dual. Let $A = G \times \hat{G}$ and let σ be the complex-valued function on $A \times A$ defined by the equation[2] $\sigma(x_1, y_1; x_2, y_2) = y_1(x_2)[y_2(x_1)]^-$. By analogy with the case in which G is a finite-dimensional vector space, it is natural to call an automorphism α of A symplectic (cf. [6]) whenever $\sigma(\alpha(x_1, y_1); \alpha(x_2, y_2)) \equiv \sigma(x_1, y_1; x_2, y_2)$. Let Σ_G denote the group of all bicontinuous symplectic automorphisms of A. In the special case in which G is the additive group of a locally compact field (of characteristic $\neq 2$) the group Σ_G plays a rôle in certain unpublished[3] number-theoretical investigations of A. Weil. In a recent[3] lecture at Harvard, Weil showed, among other things, that Σ_G in this case admits a "natural" infinite-dimensional projective unitary representation.

In the present note we shall show first (§2) that this result holds for any G for which $x \rightarrow x^2$ is an automorphism, and that the existence of the representation in question is a more or less immediate consequence of an earlier result of the author (Theorem 1 of [3]). Then in §3 we shall show that the results of our later paper [4] yield further information. In particular, we shall show that it is possible to weaken the assumption that $x \rightarrow x^2$ be an automorphism and to prove that the projective representation in question is continuous in a certain sense.

I. E. Segal, who was also present at Weil's lecture, has observed that the special case in which G is a real vector group occurs in his own work [5] and that of David Shale [7]. Shale's paper contains a deeper treatment of the representation than we shall attempt here. Segal is familiar with [3] and has independently observed that one can give the argument of §2 below. He has written a note [6] in which a more complicated method is used to obtain a stronger result. Both the method and result are different from those in §3 below.

2. The case in which $x \rightarrow x^2$ is an automorphism of G. Let U be any strongly continuous unitary σ-representation of $A = G \times \hat{G}$; that is, any strongly continuous map $x, y \rightarrow U_{x,y}$, where the $U_{x,y}$ are unitary

Received by the editors March 11, 1964.

[1] See §4.

[2] [ ]$^-$ denotes the complex conjugate.

[3] This note was written in January and February of 1963. A part of Weil's investigations have now appeared under the title *Sur certaines groupes d'operateurs unitaires*, Acta Math. 11 (1964), 145–211.

operators in a separable Hilbert space and $U_{(x_1,y_1)(x_2,y_2)}=\sigma(x_1, y_1; x_2, y_2)U_{x_1,y_1}U_{x_2,y_2}$. Let $V_x=U_{x,e}$, $W_y=U_{e,y}$. Then $U_{x,y}=V_xW_y[y(x)]^-$ and V and W are ordinary unitary representations of G and $\hat{G}$, respectively. Conversely, as a simple calculation shows, $x, y \to V_xW_y[y(x)]^-$ defines a strongly continuous unitary σ-representation of A if and only if the following "commutation relations" hold:

$$V_xW_y = y(x^2)W_yV_x. \tag{$*$}$$

If $x\to x^2$ is an automorphism, there exists, for each V, a unique strongly continuous unitary representation $\tilde{V}$ such that $\tilde{V}_{x^2}=V_x$ for all x in G. Clearly, V and W satisfy (*) if and only if $\tilde{V}$ and W satisfy

$$\tilde{V}_xW_y = y(x)W_y\tilde{V}_x. \tag{$*'$}$$

Now, according to Theorem 1 of [3], (*′) has, to within equivalence, just one irreducible solution. Here A has just one equivalence class of irreducible strongly continuous unitary σ-representations. Let U be any member of this class and for each $\alpha\in\Sigma_G$ let $U^\alpha_{x,y}=U_{\alpha(x,y)}$. Then U^α is also a member of the class and, as such, is equivalent to U. Let M_α set up the equivalence. Then $U_{\alpha(x,y)}\equiv M_\alpha U_{x,y}M_\alpha^{-1}$ and M_α is determined up to a multiplicative constant. Since $M_\alpha M_\beta$ and $M_{\alpha\beta}$ both set up an equivalence between $U^{\alpha\beta}$ and U, it follows that $\alpha\to M_\alpha$ is a projective unitary representation of Σ_G.

3. **Application of the results of [4].** Projective representations are studied systematically in [4] and one can obtain the uniqueness theorem which we have just applied, in addition to some further information, by specializing theorems of that paper. §8 of [4] contains a study of how the σ-representations of a more or less general separable locally compact group $\mathcal{G}$ are related to those of a closed normal subgroup N. (We now adapt the terminology of [4] and omit the words continuous and unitary in speaking of σ-representations with these properties.) Let us take $\mathcal{G}$ to be A and N to be $G\times e$. According to Theorem 8.4, there is a family of irreducible σ-representations of G for each orbit in a certain action of $\mathcal{G}/N$ on $\hat{N}$. In the case at hand we may identify $\mathcal{G}/N$ with $\hat{G}$ and $\hat{N}$ with $\hat{G}$. We then find that the action of y in $\hat{G}$ on y_1 in $\hat{G}$ is to take it into y_1y^2. Thus the orbits are the cosets of the subgroup S of all squares in $\hat{G}$. The equivalence classes of irreducible σ-representations associated with a fixed orbit correspond one-to-one to the equivalence classes of τ-representations of the subgroup of $\mathcal{G}/N$ leaving a point of the orbit fixed. Here τ is a multiplier described in the proof of Theorem 8.2.

Suppose that $x\to x^2$ is an automorphism of G so that $y\to y^2$ is an automorphism of $\hat{G}$. Then $S=\hat{G}$ and there is only one orbit. Moreover, the relevant subgroup of $\mathcal{G}/N=\hat{G}$ is the identity so there can be only one equivalence class of τ-representations. Since S is closed, Theorem 9.1 applies and tells us that there are no irreducible σ-representations except those described by Theorem 8.4. Thus A has only one equivalence class of σ-representations whenever $x\rightarrow x^2$ is an automorphism of G.

To see what §§8 and 9 of [4] tell us in more general cases, let us first observe that the multiplier τ is identically one whenever $\mathcal{G}=A$, $N=G\times e$ and σ is as above. It follows that the equivalence classes of irreducible σ-representations associated with each S coset correspond one-to-one to the members of the character group $\hat{S}'$ of the group S' of all elements of order 2 in $\hat{G}$. When S is closed, it follows from Theorem 9.1 that the irreducible σ-representations described in Theorem 8.4 are exhaustive. Though we shall not do so here, the considerations of §5 of [4] can be extended to prove that A has further irreducible σ-representations whenever S is not closed. Thus we have

THEOREM 1. *The group $A=G\times\hat{G}$ has just one equivalence class of irreducible σ-representations if and only if $x\rightarrow x^2$ is an automorphism of G. If $x\rightarrow x^2$ is not an automorphism of G then there is a "natural" one-to-one map of $(\hat{G}/S)\times\hat{S}'$ into the set of equivalence classes of irreducible σ-representations of A. Here, S is the group of all squares in $\hat{G}$ and S' is the group of all elements of order 2 in $\hat{G}$. This map is "onto" if and only if S is closed in $\hat{G}$.*

We now reformulate the one-to-one correspondence described in Theorem 1 in such a way that the natural action of Σ_G on the irreducible σ-representations of A can be studied. The notion of "induced representation" is defined for σ-representations on p. 274 of [4] and, by Theorem 8.4 of that paper, the σ-representations of A described in Theorem 1 above are all induced by one-dimensional σ-representations of the subgroup $G\times S'$ of A. The mapping $x,\ \eta\rightarrow\sigma(x,\ e;\ e,\ \eta)=[\eta(x)]^-=\eta(x)$ is one such and the most general one is $x,\ \eta\rightarrow\eta(x)y_1(x)z_1(\eta)$, where $y_1\in\hat{G}_1$ and $z_1\in\hat{S}'$. Let U^{y_1,z_1} denote the σ-representation of A induced by $x,\ \eta\rightarrow\eta(x)y_1(x)z(\eta)$. By Theorem 8.4, U^{y_1,z_1} is irreducible and U^{y_1,z_1} is equivalent to U^{y_2,z_2} if and only if y_1 and y_2 lie in the same S coset. The resulting one-to-one map of $\hat{G}/S\times\hat{S}'$ into irreducible σ-representations is the one alluded to in Theorem 1. Let $S^\perp$ denote the annihilator of S in G, that is, the group of all elements of order 2 in G. Then $S^\perp\times S'$ is just the subgroup of all elements of order 2 in A and as such can be described without reference to the factorization $A=G\times\hat{G}$. The restriction to $S^\perp\times S'$ of the

σ-representation U^{y_1,z_1} is easily computed to be a multiple of the one-dimensional σ-representation ξ, $\eta \rightarrow \eta(\xi)y_1(\xi)z_1(\eta)$. Moreover, it follows from Theorem 7.1 of [4] that any irreducible σ-representation of A which is not equivalent to one of the U^{y_1,z_1} must have a restriction to $S^\perp \times S'$ which is a direct integral of infinitely many distinct one-dimensional σ-representations. We can now conclude the truth of the following variant of Theorem 1.

THEOREM 1'. *Let A_0 be the subgroup of $A = G \times \hat{G}$ consisting of all elements of A of order 2. Then, for each member $\mathfrak{X}$ of $\hat{A}_0$, there exists (to within equivalence) just one irreducible σ-representation W^{X} of A whose restriction to A_0 is a multiple of the one-dimensional σ-representation $\xi, \eta \rightarrow \eta(\xi)\mathrm{X}(\xi, \eta)$. The σ-representations W^{X} will exhaust the irreducible σ-representations of A if and only if S is a closed subgroup of $\hat{G}$.*

It is clear that each $\alpha \in \Sigma_G$ carries A_0 onto A_0 and every irreducible σ-representation of the form W^{X} into another of the same form $W^{\mathrm{X}}_{\alpha(x,y)} \equiv W^{\mathrm{X}\alpha}_{x,y}$. However, it is not clear that the mapping $\mathrm{X} \rightarrow \mathrm{X}^\alpha$ is that induced by the action of α on A_0. Let $\phi_0(\xi, \eta) = \eta(\xi)$. Then $\mathrm{X}^\alpha(\xi, \eta) = \mathrm{X}(\alpha(\xi, \eta))\phi_0(\alpha(\xi, \eta))/\phi_0(\xi, \eta)$. Thus, if we restrict α to lie in the subgroup Σ_G^0 of Σ_G consisting of all α with $\phi_0(\alpha(\xi, \eta)) \equiv \phi_0(\xi, \eta)$, then $\mathrm{X}^\alpha(\xi, \eta) \equiv \mathrm{X}(\alpha(\xi, \eta))$ and, taking $\mathfrak{X}$ to be the identity character, we conclude the truth of

THEOREM 2. *Let W be the unique irreducible σ-representation of $A = G \times \hat{G}$ which is a multiple of the one-dimensional σ-representation $\xi, \eta \rightarrow \eta(\xi)$ when restricted to $A_0 = S^\perp \times S'$. Then, for every α in Σ_G^0, the σ-representation $x, y \rightarrow W_{\alpha(x,y)}$ of A is equivalent to W.*

If we replace the unique σ-representation of §2 by W and replace Σ_G by Σ_G^0 we may define a projective representation M just as we did at the end of §2. However, we now need make no restriction on the endomorphism $x \rightarrow x^2$. Note that $\Sigma_G^0 = \Sigma_G$ whenever either $S^\perp$ or S' reduces to the identity.

We complete this section with the promised result about continuity.

THEOREM 3. *Let Σ be a subgroup of Σ_G^0 topologized so as to be separable, locally compact and such that $x, y, \alpha \rightarrow \alpha(x, y)$ is continuous from $A \times \Sigma$ to A. Then the projective representation M of Σ_G^0, defined as indicated above, is continuous when restricted to Σ in the sense that $\alpha \rightarrow |M_\alpha(\phi) \cdot \psi|$ is a continuous function of α for all ϕ and ψ in the Hilbert space.*

PROOF. Let $\mathcal{G}$ be the group of all triples x, y, α with $x \in G$, $y \in \hat{G}$,

$\alpha \in \Sigma$. Make $\mathcal{G}$ into a separable locally compact group by setting $(x_1, y_1, \alpha_1)(x_2, y_2, \alpha_2) = ((x_1, y_1)\alpha(x_2, y_2), \alpha_1\alpha_2)$. Let W be the irreducible σ-representation of $G \times \hat{G}$ described in Theorem 2. Let σ' be the multiplier for G defined as follows.

$$\sigma'(x_1, y_1, \alpha_1; x_2, y_2, \alpha_2) = \sigma(x_1, y_1, \alpha_1(x_2, y_2)).$$

Apply Theorem 8.2 of [4] with $\mathcal{K} = G \times \hat{G} \times e$, $L = W$. Then our M is just the restriction to $e \times \Sigma$ of the M of that theorem. As such it is continuous in the sense indicated.

4. **On avoiding the hypothesis of separability.** Loomis in [1] has generalized the main theorem of [3] to the inseparable case. Using this generalization we see at once that the discussion of §2 does not really require the separability hypothesis. The extent to which separability may be avoided in §3 is not clear. Loomis's paper [2] shows that one of the key results of [4] is valid for nonseparable groups but a close examination of the arguments of [4] would have to be made before one could say whether or not separability is needed elsewhere.

Bibliography

1. L. H. Loomis, *Note on a theorem of Mackey*, Duke Math. J. **19** (1952), 641–645.

2. ———, *Positive definite functions and induced representations*, Duke Math. J. **27** (1960), 564–579.

3. G. W. Mackey, *On a theorem of Stone and von Neumann*, Duke Math. J. **16** (1949), 313–326.

4. ———, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311.

5. I. E. Segal, *Foundations of the theory of dynamical systems of infinitely many degrees of freedom*. I, Mat. Fys. Medd. Danske Vid. Selsk. **31** (1959), no. 12, 1–39.

6. ———, *Transforms for operators and symplectic automorphisms over a locally compact abelian group*, Math. Scand. **13** (1963), 31–43.

7. D. Shale, *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. **103** (1962), 149–167.

Harvard University