ON INTEGRAL TRANSFORMATIONS WITH
POSITIVE KERNEL

EMILIO GAGLIARDO

1. Let $X$, $Y$ be two totally $\sigma$-finite measure spaces and let $F_X$, $F_Y$ be the classes of measurable functions defined, and finite a.e., in $X$, $Y$, respectively.

Let $K(x, y) \geq 0$ be a measurable non-negative function on $X \times Y$. Let us denote by $T$, $T^*$ the transformations:

$$(Tu)(y) = \int_X K(x, y)u(x)dx, \quad (T^*v)(x) = \int_Y K(x, y)v(y)dy,$$

the domain of $T$ being the set of the functions $u(x) \in F_X$ such that the first integral exists and is finite for almost all $y$, and the domain of $T^*$ being analogously defined.

We are interested in the study of necessary and sufficient conditions in order that $T$ restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leq c_0$.

Obviously such a necessary and sufficient condition is given by:

$$\int\int_{X \times Y} K(x, y)u(x)v(y)\, dx\, dy \leq c_0 \|u\|_{L^p(X)}\|v\|_{L^q(Y)},$$

for all $u \in L^p(X)$, $v \in L^q(Y)$.

In the present paper we will prove the necessity of the following sufficient condition:

[1.1] Theorem (N. Aronszajn). Let $1 < q \leq p < +\infty$, $1/p' + 1/p = 1/q' + 1/q = 1$. A sufficient condition in order that $T$ restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leq c_0$ is that for every $\epsilon > 0$ there exist two measurable functions $\phi(x)$, $\psi(y)$, positive and finite a.e., such that:

$$(T\phi)(y) \leq (c_0 + \epsilon)(\psi(y))^{q'/q}, \quad (T^*\psi)(x) \leq (c_0 + \epsilon)(\phi(x))^{p'/p},$$

Presented to the Society, January 24, 1963 under the title On bounded integral transformations, with positive kernel; received by the editors July 30, 1963, and, in revised form, October 11, 1963.

1 Paper written under Research Grant GP-439 with the National Science Foundation, University of Kansas.

2 See [1, p. 36]; also more particular results of I. Schur, G. H. Hardy, J. E. Littlewood, G. Pólya [3, p. 227], and of Holmgren (see T. Carleman [2, p. 112]). For the sake of completeness we give here also the proof of this theorem.
no other condition for \( p = q \),

\[
(1.3) \quad \int_X \int_Y K(x, y)\phi(x)\psi(y) \, dx \, dy \leq c_0 + \varepsilon \quad \text{for } p \neq q.
\]

In §3 we will prove the following:

[1.II] Theorem. Let \( 1 < p < +\infty, 1 < q < +\infty \).

A necessary condition in order that \( T \) restricted to \( L^p(X) \) be a bounded transformation of \( L^p(X) \) into \( L^q(Y) \) with bound \( \leq c_0 \) is given by (1.2) with \( \phi, \psi \), positive and finite a.e., satisfying:

\[
(1.4) \quad \|\phi\|_{L^p(X)} \leq 1, \quad \|\psi\|_{L^q(Y)} \leq 1.
\]

This theorem in the case \( p = q = 2 \) is due to S. Karlin [4].

[1.III] Corollary. Obviously (1.2)–(1.3) is weaker than (1.2)–(1.4), and, consequently, for \( q \leq p \) both (1.2)–(1.3) and (1.2)–(1.4), a posteriori equivalent, are necessary and sufficient.

[1.IV] Remark. For \( q > p \), as we will show by an example, even the stronger condition (1.2)–(1.4) is in general not sufficient.

[1.V] Remark. For \( \varepsilon = 0 \) Theorem [1.II] does not hold.

[1.VI] Remarks. For \( q = 1 \), \( p \) finite or not, a necessary and sufficient condition is: \( \|\int_K(x, y)\, dy\|_{L^p(X)} \leq c_0 \). For \( p = +\infty \), \( q \) finite or not, a necessary and sufficient condition is: \( \|\int_K(x, y)\, dx\|_{L^q(Y)} \leq c_0 \).

2. Proof of Theorem [1.I].

Case \( p \neq q \). Given \( u \in L^p(X), v \in L^q(Y) \), we note that:

\[
\left| \int_X \int_Y K(x, y)u(x)v(y) \, dx \, dy \right|
\]

\[= \left| \int_X \int_Y \frac{K(x, y)\phi(x)\psi(y)}{\psi^{1/q'}} \, dx \, dy \right|
\]

\[\leq \left( \int_X \int_Y K(x, y)\phi(x)\psi(y) \, dx \, dy \right)^{1/q-1/p}
\]

\[\cdot \left( \int_X \int_Y K(x, y) \frac{\psi}{\phi^{1/p'}} \, dx \, dy \right)^{1/p'}
\]

\[\cdot \left( \int_X \int_Y K(x, y) \frac{\phi}{\psi^{1/q'}} \, dx \, dy \right)^{1/q'}
\]

\[\leq (c_0 + \varepsilon)\|u\|_{L^p(X)}\|v\|_{L^q(Y)}.
\]

\footnote{In the following meaning: functions \( \phi, \psi \) satisfying (1.2), (1.4) satisfy also (1.3), and conversely if there exist \( \phi, \psi \) satisfying (1.2), (1.3) then there exist also \( \phi, \psi \)(may be different) satisfying (1.2), (1.4).}
Case \( p = q \). Since \( \frac{1}{q} - \frac{1}{p} = 0 \) the proof here given does not require the boundedness of the integral in (1.3).

3. Proof of Theorem [1.11].

[3.1] Lemma. Let \( B \) be a Banach space and \( P \) a convex cone in \( B \). By calling this cone “positive,” \( B \) will be taken as an ordered Banach space. Let us suppose for \( B \) and \( P \) that every bounded increasing sequence in \( P \) converges, more precisely:

\[
\{ u_n \} \subset P, \quad u_{n+1} - u_n \in P, \quad \| u_n \| \leq M < +\infty \Rightarrow u_n \to u \in P.
\]

Let \( S \) be a (not necessarily linear) transformation defined in \( P \) such that

\[
S(P) \subset P,
\]

\[
S \text{ is nondecreasing: } u, v, v - u \in P \Rightarrow Sv - Su \in P,
\]

\[
S \text{ is continuous},^4 \quad \| u \| \leq 1 \Rightarrow \| Su \| \leq 1.
\]

Then for every \( \sigma > 0 \) there is \( u \in P \) such that:

\[
(1 + \sigma)u - Su \in P,
\]

\[
\| u \| \leq 1, \quad u \neq 0.
\]

Proof of lemma. Choose \( p \in P, \quad p \neq 0, \quad \| p \| \leq \sigma/(1 + \sigma) \) and consider the sequence \( \{ u_n \} \subset P \) defined as follows:

\[
u_1 = p, \quad u_n = p + \frac{1}{1 + \sigma} Su_{n-1}.
\]

Obviously by induction,

\[
\| u_n \| \leq 1, \quad u_{n+1} - u_n \in P.
\]

So by (3.1), and since \( S \) is continuous:

\[
u_n \to u \in P, \quad \| u \| \leq 1, \quad u = p + \frac{1}{1 + \sigma} Su.
\]

Hence also:

\[
u \neq 0, \quad (1 + \sigma)u - Su = (1 + \sigma)p \in P.
\]

The necessity of (1.2), (1.4) follows from Lemma [3.1] by taking \( B = L^p(X), \quad u \in P \Leftrightarrow u(x) \geq 0 \) a.e.,

\[\quad^4\text{In the strong topology, or, more generally, relative to any topology for which (3.1) holds, and the norm is lower semicontinuous.}\]
$$S(u) = \left( \frac{1}{c_0 + \epsilon} T^* \left( v_0 + \frac{1}{c_0 + \epsilon} T \left( u_0 + \frac{1}{1 + \sigma} u \right) \right) \right)^{\frac{q / q'}{p'/p}},$$

where

$$v_0(y) > 0 \text{ a.e., } \|v_0\|_{L^p} \leq \frac{1}{c_0 + \epsilon},$$

$$u_0(x) > 0 \text{ a.e., } \|u_0\|_{L^p} \leq \frac{\sigma}{1 + \sigma},$$

and $\sigma$ is such that

$$c_0(1 + \sigma)^{\frac{2p}{p'}} \leq c_0 + \epsilon.$$

For, in the present case, (3.3) becomes

$$\left( \frac{1}{c_0 + \epsilon} T^* \left( v_0 + \frac{1}{c_0 + \epsilon} T \left( u_0 + \frac{1}{1 + \sigma} u \right) \right) \right)^{\frac{q / q'}{p'/p}} \leq (1 + \sigma)u$$

$$\|u\|_{L^p(x)} \leq 1,$$

and by putting

$$\phi = u_0 + \frac{1}{1 + \sigma} u, \quad \psi = \left( v_0 + \frac{1}{c_0 + \epsilon} T \phi \right)^{\frac{q / q'}{p'/p}},$$

we note that:

$$T\phi = (c_0 + \epsilon)\psi^{q' / q} - v_0 \leq (c_0 + \epsilon)\psi^{q' / q},$$

$$T^*\psi \leq c_0((1 + \sigma)u)^{p' / p'} = c_0(1 + \sigma)^{\frac{2p}{p'}}(\phi - u_0)^{p' / p'} \leq (c_0 + \epsilon)\phi^{p'/p'},$$

$$\|\phi\|_{L^p(x)} \leq \frac{\sigma}{1 + \sigma} + \frac{1}{1 + \sigma} = 1, \quad \phi \geq u_0 > 0 \text{ a.e.,}$$

$$\|\psi\|_{L^{p'}(Y)} \leq \left( \frac{\epsilon}{c_0 + \epsilon} + \frac{c_0}{c_0 + \epsilon} \right)^{\frac{q / q'}{p'/p'}} = 1, \quad \psi \geq (v_0)^{q / q'} > 0 \text{ a.e.}$$

4. Proof of Remark [1.IV]. Consider the following example:

$$p = 2, \quad (p' = 2), \quad q = 4, \quad (q' = \frac{q}{q'}), \quad X = (0, +\infty), \quad Y = (0, 1),$$

$$K(x, y) = \begin{cases} 
\frac{y}{2} & \text{for } \frac{y}{2} < x < y, \\
0 & \text{elsewhere},
\end{cases}$$
To prove the first inequality in (1.2) (with some suitable $c_0 + \varepsilon$) it is enough to note that $(T\phi)(y)$ is bounded, and $\psi(y)$ has a positive infimum, and to prove the second inequality it is enough to note that $(T^*\psi)(x)$ is bounded, and is $\equiv 0$ for $x \geq 1$.

Conditions (1.4) are immediately verified.

Finally, to prove that $T$ is not, however, a bounded transformation of $L^2(0, + \infty)$ into $L^4(0, 1)$ it is enough to consider the function:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ x^{1/4} & \text{elsewhere,} \end{cases}$$

which belongs to $L^2(0, + \infty)$, while $(Tf)(y) \notin L^4(0, 1)$.

5. Proof of Remark [1.V]. Let us take $p = q = p' = q' = 2$, $X = Y = x$-axis,

$$K(x, y) = K(y, x) = \begin{cases} 1 & \text{for } |x - y| \leq \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases} c_0 = ||T|| = 1.$$  

We will prove that there are no non-negative $\phi, \psi, \phi(x) \neq 0 \neq \psi(x)$, such that $\phi \in L^2(X)$, $T\phi \leq \psi$, $T\psi \leq \phi$.

Let us suppose that such $\phi, \psi$ exist. Putting $u(x) = \phi(x) + \psi(x)$, we have

$$u(x) \geq 0, \quad u(x) \neq 0, \quad Tu \leq u, \quad Tu \leq T\phi + \psi \in L^2.$$  

First case. Let us suppose $Tu = u$ a.e. This case can be easily excluded (for instance by taking Fourier transforms of both sides.)

Second case. $Tu < u$ on a set of positive measure. Then we can choose $c_1, \alpha_1$ such that:

$$\int_{-\alpha}^{\alpha} u(x) \, dx - \int_{-\alpha}^{\alpha} Tu \, dx \geq c_1 > 0 \quad \text{for } \alpha \geq \alpha_1.$$  

On the other hand since $Tu \in L^2$ there is a set $I$ of infinite measure such that

$$\int_{-\alpha - 1/2}^{-\alpha + 1/2} u(x) \, dx + \int_{\alpha - 1/2}^{\alpha + 1/2} u(x) \, dx = (Tu)(-\alpha) + (Tu)(\alpha) < c_1 \quad \text{for } \alpha \in I.$$
From the definition of $T$ it follows that for every $\alpha > 0$,

$$\int_{-a}^{a} u(x) \, dx - \int_{-a}^{a} Tu \, dx \leq \int_{-a-1/2}^{a+1/2} u(x) \, dx + \int_{a-1/2}^{a+1/2} u(x) \, dx,$$

and if we take $\alpha \geq \alpha_1$, $\alpha \in I$ we have a contradiction.

6. Proof of Remarks [1.VI].

First Remark: Sufficiency. Given $u \in L^p(X)$, $v \in L^\infty(Y)$ we have

$$\left| \int_{X} \int_{Y} K(x, y)u(x)v(y) \, dxdy \right| \leq \|v\|_{L^\infty(Y)} \int_{X} \int_{Y} |K(x, y)| \, u(x) \, dxdy \leq \|v\|_{L^\infty(Y)} \int_{Y} |K(x, y)| \, dy \left\|u\right\|_{L^p(X)} \leq c_0 \|v\|_{L^\infty(Y)} \|u\|_{L^p(X)}.$$

Necessity. Since for every $u \in L^p(X)$, $v \in L^\infty(Y)$ we have

$$\int_{X} \int_{Y} K(x, y)u(x)v(y) \, dxdy \leq c_0 \|u\|_{L^p(X)} \|v\|_{L^\infty(Y)}.$$

taking $v \equiv 1$ we have

$$\int_{X} u(x) \left( \int_{Y} K(x, y) \, dy \right) \, dx \leq c_0 \|u\|_{L^p(X)}$$

and since $u \in L^p(X)$ is still arbitrary:

$$\left\| \int_{Y} K(x, y) \, dy \right\|_{L^p(X)} \leq c_0.$$

Second Remark. The proof is analogous.

References

2. T. Carleman, Sur les équations intégrales singulières à noyau réel et symétrique, Upsala, 1923.

University of Kansas