ON INTEGRAL TRANSFORMATIONS WITH POSITIVE KERNEL[1]

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1. Let X, Y be two totally σ-finite measure spaces and let F_X, F_Y be the classes of measurable functions defined, and finite a.e., in X, Y, respectively.

Let $K(x, y) \geqq 0$ be a measurable non-negative function on $X \times Y$.

Let us denote by T, T^* the transformations:

$$(Tu)(y) = \int_X K(x, y)u(x)dx, \qquad (T^*v)(x) = \int_Y K(x, y)v(y)dy,$$

the domain of T being the set of the functions $u(x) \in F_X$ such that the first integral exists and is finite for almost all y, and the domain of T^* being analogously defined.

We are interested in the study of necessary and sufficient conditions in order that T restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leqq c_0$.

Obviously such a necessary and sufficient condition is given by:

$$\iint_{X\times Y} K(x, y)u(x)v(y)\, dxdy \leqq c_0 \| u \|_{L^p(X)} \| v \|_{L^{q'}(Y)}, \tag{1.1}$$

$$\forall u \in L^p(X), \ \forall v \in L^{q'}(Y).$$

In the present paper we will prove the necessity of the following sufficient condition:

[1.I] THEOREM (N. ARONSZAJN[2]). *Let* $1 < q \leqq p < +\infty$, $1/p' + 1/p = 1/q' + 1/q = 1$. *A sufficient condition in order that T restricted to $L^p(X)$ be a bounded transformation of $L^p(X)$ into $L^q(Y)$ with bound $\leqq c_0$ is that for every $\epsilon > 0$ there exist two measurable functions $\phi(x)$, $\psi(y)$, positive and finite a.e., such that*:

$$\begin{aligned} (T\phi)(y) &\leqq (c_0 + \epsilon)(\psi(y))^{q'/q}, \\ (T^*\psi)(x) &\leqq (c_0 + \epsilon)(\phi(x))^{p/p'}, \end{aligned} \tag{1.2}$$

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[2] See [1, p. 36]; also more particular results of I. Schur, G. H. Hardy, J. E. Littlewood, G. Pólya [3, p. 227], and of Holmgren (see T. Carleman [2, p. 112]). For the sake of completeness we give here also the proof of this theorem.

no other condition for $p = q$,

$$\iint_{X\times Y} K(x, y)\phi(x)\psi(y)\, dxdy \leqq c_0 + \epsilon \qquad \textit{for } p \neq q. \tag{1.3}$$

In §3 we will prove the following:

[1.II] Theorem. *Let* $1 < p < +\infty$, $1 < q < +\infty$.

A necessary condition in order that T restricted to $L^p(X)$ *be a bounded transformation of* $L^p(X)$ *into* $L^q(Y)$ *with bound* $\leqq c_0$ *is given by* (1.2) *with* ϕ, ψ, *positive and finite a.e., satisfying*:

$$\|\phi\|_{L^p(X)} \leqq 1, \qquad \|\psi\|_{L^{q'}(Y)} \leqq 1. \tag{1.4}$$

This theorem in the case $p = q = 2$ is due to S. Karlin [4].

[1.III] Corollary. *Obviously* (1.2)–(1.3) *is weaker than* (1.2)–(1.4), *and, consequently, for* $q \leqq p$ *both* (1.2)–(1.3) *and* (1.2)–(1.4), *a posteriori equivalent,*[3] *are necessary and sufficient.*

[1.IV] Remark. *For* $q > p$, *as we will show by an example, even the stronger condition* (1.2)–(1.4) *is in general not sufficient.*

[1.V] Remark. *For* $\epsilon = 0$ *Theorem* [1.II] *does not hold.*

[1.VI] Remarks. *For* $q = 1$, p *finite or not, a necessary and sufficient condition is*: $\|\int_Y K(x, y)\, dy\|_{L^{p'}(X)} \leqq c_0$. *For* $p = +\infty$, q *finite or not, a necessary and sufficient condition is*: $\|\int_X K(x, y)\, dx\|_{L^q(Y)} \leqq c_0$.

2. Proof of Theorem [1.I].

Case $p \neq q$. Given $u \in L^p(X)$, $v \in L^{q'}(Y)$, we note that:

$$\begin{aligned}
&\left|\int_X\int_Y K(x, y)u(x)v(y)\, dxdy\right| \\
&\quad = \left|\int_X\int_Y K^{1/q-1/p}\phi^{1/q-1/p}\psi^{1/q-1/p}K^{1/p}\frac{\psi^{1/p}}{\phi^{1/p'}}\, uK^{1/q'}\frac{\phi^{1/q'}}{\psi^{1/q}}\, v\, dxdy\right| \\
&\quad \leqq \left(\int_X\int_Y K(x, y)\phi(x)\psi(y)\, dxdy\right)^{1/q-1/p} \\
&\quad\quad \cdot\left(\int_X\int_Y K(x, y)\frac{\psi}{\phi^{p/p'}}\, |u|^p\, dxdy\right)^{1/p} \\
&\quad\quad \cdot\left(\int_X\int_Y K(x, y)\frac{\phi}{\psi^{q'/q}}\, |v|^{q'} dxdy\right)^{1/q'} \\
&\quad \leqq (c_0 + \epsilon)\|u\|_{L^p(X)}\|v\|_{L^{q'}(Y)}.
\end{aligned}$$

[3] In the following meaning: functions ϕ, ψ satisfying (1.2), (1.4) satisfy also (1.3), and conversely if there exist ϕ, ψ satisfying (1.2), (1.3) then there exist also $\bar{\phi}$, $\bar{\psi}$ (may be different) satisfying (1.2), (1.4).

Case $p=q$. Since $1/q-1/p=0$ the proof here given does not require the boundedness of the integral in (1.3).

3. **Proof of Theorem [1.II].**

[3.I] LEMMA. *Let B be a Banach space and P a convex cone in B. By calling this cone "positive," B will be taken as an ordered Banach space. Let us suppose for B and P that every bounded increasing sequence in P converges, more precisely*:

$$\{u_n\} \subset P, \quad u_{n+1} - u_n \in P, \quad \|u_n\| \leqq M < +\infty \Rightarrow u_n \to u \in P. \tag{3.1}$$

Let S be a (not necessarily linear) transformation defined in P such that

$$\begin{aligned} &S(P) \subset P, \\ &S \text{ is nondecreasing: } u, v, v-u \in P \Rightarrow Sv - Su \in P, \\ &S \text{ is continuous,}^4 \\ &\|u\| \leqq 1 \Rightarrow \|Su\| \leqq 1. \end{aligned} \tag{3.2}$$

Then for every $\sigma>0$ there is $u\in P$ such that:

$$\begin{aligned} &(1+\sigma)u - Su \in P, \\ &\|u\| \leqq 1, \qquad u \neq 0. \end{aligned} \tag{3.3}$$

PROOF OF LEMMA. Choose $p\in P$, $p\neq 0$, $\|p\|\leqq\sigma/(1+\sigma)$ and consider the sequence $\{u_n\}\subset P$ defined as follows:

$$u_1 = p, \qquad u_n = p + \frac{1}{1+\sigma} Su_{n-1}.$$

Obviously by induction,

$$\|u_n\| \leqq 1, \qquad u_{n+1} - u_n \in P.$$

So by (3.1), and since S is continuous:

$$u_n \to u \in P, \qquad \|u\| \leqq 1, \qquad u = p + \frac{1}{1+\sigma} Su.$$

Hence also:

$$u \neq 0, \qquad (1+\sigma)u - Su = (1+\sigma)p \in P.$$

The necessity of (1.2), (1.4) follows from Lemma [3.I] by taking $B=L^p(X)$, $u\in P\Leftrightarrow u(x)\geqq 0$ a.e.,

[4] In the strong topology, or, more generally, relative to any topology for which (3.1) holds, and the norm is lower semicontinuous.

$$S(u) = \left(\frac{1}{c_0} T^*\left(v_0 + \frac{1}{c_0+\epsilon} T\left(u_0 + \frac{1}{1+\sigma} u\right)\right)^{q/q'}\right)^{p'/p},$$

where

$$v_0(y) > 0 \text{ a.e.}, \qquad \|v_0\|_{L^q} \leqq \frac{\epsilon}{c_0+\epsilon},$$

$$u_0(x) > 0 \text{ a.e.}, \qquad \|u_0\|_{L^p} \leqq \frac{\sigma}{1+\sigma},$$

and σ is such that

$$c_0(1+\sigma)^{2p/p'} \leqq c_0 + \epsilon.$$

For, in the present case, (3.3) becomes

$$\left(\frac{1}{c_0} T^*\left(v_0 + \frac{1}{c_0+\epsilon} T\left(u_0 + \frac{1}{1+\sigma} u\right)\right)^{q/q'}\right)^{p'/p} \leqq (1+\sigma)u$$

$$\|u\|_{L^p(X)} \leqq 1,$$

and by putting

$$\phi = u_0 + \frac{1}{1+\sigma} u, \qquad \psi = \left(v_0 + \frac{1}{c_0+\epsilon} T\phi\right)^{q/q'},$$

we note that:

$$T\phi = (c_0+\epsilon)(\psi^{q'/q} - v_0) \leqq (c_0+\epsilon)\psi^{q'/q},$$

$$T^*\psi \leqq c_0((1+\sigma)u)^{p/p'} = c_0(1+\sigma)^{2p/p'}(\phi - u_0)^{p/p'} \leqq (c_0+\epsilon)\phi^{p/p'},$$

$$\|\phi\|_{L^p(X)} \leqq \frac{\sigma}{1+\sigma} + \frac{1}{1+\sigma} = 1, \qquad \phi \geqq u_0 > 0 \text{ a.e.},$$

$$\|\psi\|_{L^{q'}(Y)} \leqq \left(\frac{\epsilon}{c_0+\epsilon} + \frac{c_0}{c_0+\epsilon}\right)^{q/q'} = 1, \qquad \psi \geqq (v_0)^{q/q'} > 0 \text{ a.e.}$$

4. **Proof of Remark [1.IV].** Consider the following example:

$$p = 2, \quad (p' = 2), \quad q = 4, \quad (q' = \tfrac{4}{3}), \quad X = (0, +\infty), \quad Y = (0, 1),$$

$$K(x, y) = \begin{cases} \dfrac{1}{y} & \text{for } \dfrac{y}{2} < x < y, \\ 0 & \text{elsewhere,} \end{cases}$$

$$\phi(x) = \frac{1}{\sqrt{2}} \frac{1}{(x+1)^{3/4}} \qquad (0 < x < +\infty),$$

$$\psi(y) = \frac{1}{\sqrt{2}} \frac{1}{(y+1)^{3/4}} \qquad (0 < y < 1).$$

To prove the first inequality in (1.2) (with some suitable $c_0+\epsilon$) it is enough to note that $(T\phi)(y)$ is bounded, and $\psi(y)$ has a positive infimum, and to prove the second inequality it is enough to note that $(T^*\psi)(x)$ is bounded, and is $\equiv 0$ for $x \geqq 1$.

Conditions (1.4) are immediately verified.

Finally, to prove that T is not, however, a bounded transformation of $L^2(0, +\infty)$ into $L^4(0, 1)$ it is enough to consider the function:

$$f(x) = \begin{cases} \dfrac{1}{x^{1/4}} & \text{for } 0 < x < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

which belongs to $L^2(0, +\infty)$, while $(Tf)(y) \notin L^4(0, 1)$.

5. **Proof of Remark** [**1.V**]. Let us take $p=q=p'=q'=2$, $X=Y$ $=$ x-axis,

$$K(x, y) = K(y, x) = \begin{cases} 1 & \text{for } |x-y| \leqq \frac{1}{2}, \\ 0 & \text{elsewhere,} \end{cases} \qquad c_0 = \|T\| = 1.$$

We will prove that there are no non-negative ϕ, ψ, $\phi(x) \not\equiv 0 \not\equiv \psi(x)$, such that $\phi \in L^2(X)$, $T\phi \leqq \psi$, $T\psi \leqq \phi$.

Let us suppose that such ϕ, ψ exist. Putting $u(x)=\phi(x)+\psi(x)$, we have

$$u(x) \geqq 0, \qquad u(x) \not\equiv 0, \qquad Tu \leqq u, \qquad Tu \leqq T\phi + \phi \in L^2.$$

First case. Let us suppose $Tu=u$ a.e. This case can be easily excluded (for instance by taking Fourier transforms of both sides.)

Second case. $Tu<u$ on a set of positive measure. Then we can choose c_1, α_1 such that:

$$\int_{-\alpha}^{\alpha} u(x)\,dx - \int_{-\alpha}^{\alpha} Tu\,dx \geqq c_1 > 0 \qquad \text{for } \alpha \geqq \alpha_1.$$

On the other hand since $Tu \in L^2$ there is a set I of infinite measure such that

$$\int_{-\alpha-1/2}^{-\alpha+1/2} u(x)\,dx + \int_{\alpha-1/2}^{\alpha+1/2} u(x)\,dx = (Tu)(-\alpha) + (Tu)(\alpha) < c_1 \qquad \text{for } \alpha \in I.$$

From the definition of T it follows that for every $\alpha > 0$,

$$\int_{-\alpha}^{\alpha} u(x)\,dx - \int_{-\alpha}^{\alpha} Tu\,dx \leqq \int_{-\alpha-1/2}^{-\alpha+1/2} u(x)\,dx + \int_{\alpha-1/2}^{\alpha+1/2} u(x)\,dx,$$

and if we take $\alpha \geqq \alpha_1$, $\alpha \in I$ we have a contradiction.

6. Proof of Remarks [1.VI].

First Remark: Sufficiency. Given $u \in L^p(X)$, $v \in L^\infty(Y)$ we have

$$\begin{aligned}\left| \int_X \int_Y K(x,y)u(x)v(y)\,dxdy \right| &\leqq \|v\|_{L^\infty(Y)} \int_X \int_Y K(x,y)\,|\,u(x)\,|\,dxdy \\ &\leqq \|v\|_{L^\infty(Y)} \left\| \int_Y K(x,y)dy \right\|_{L^{p'}(X)} \|u\|_{L^p(X)} \\ &\leqq c_0 \|u\|_{L^p(X)} \|v\|_{L^\infty(Y)}.\end{aligned}$$

Necessity. Since for every $u \in L^p(X)$, $v \in L^\infty(Y)$ we have

$$\int_X \int_Y K(x,y)u(x)v(y)\,dxdy \leqq c_0 \|u\|_{L^p(X)} \|v\|_{L^\infty(Y)},$$

taking $v \equiv 1$ we have

$$\int_X u(x) \left(\int_Y K(x,y)\,dy \right) dx \leqq c_0 \|u\|_{L^p(X)},$$

and since $u \in L^p(X)$ is still arbitrary:

$$\left\| \int_Y K(x,y)\,dy \right\|_{L^{p'}(X)} \leqq c_0.$$

Second Remark. The proof is analogous.

References

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