A SHORT PROOF OF THE JAMES PERIODICITY OF $\pi_{k+p}(V_{k+m,m})$[1]

MARK MAHOWALD

Let $V_{k+m,m}$ be the Stiefel manifold of m-frames in $k+m$ Euclidean space. The purpose of this note is to give a short proof of the following theorem of James [2].

THEOREM. *If $p<k-1$ and $m<k$, then $\pi_{k+p}(V_{k+m,m})$ is periodic in k of period $2^{\phi(m-1)}$, where $\phi(n)$ is the number of integers s such that $s=0, 1, 2$ or $4 \bmod 8$ and $0<s\leqq n$.*

REMARK. Since $\pi_{k+p}(V_{k+m,m})\cong\pi_{k+p}(V_{k+p+2,p+2})$ for $m\geqq p+2$, the periodicity theorem can be sharpened to read $2^{\phi(p+1)}$ if $p+2\leqq m$.

PROOF. It is known [3] that the $(2k-1)$-skeleton of $V_{k+m,m}$ is of the same homotopy type as that of $P^k_{k+m-1}=P_{k+m-1}/P_{k-1}$, where P_n is the real n-dimensional projection space. On the other hand, P^k_{k+m-1} is the Thom complex of kH_{m-1} (which we will write as $T(kH_{m-1})$), where H_{m-1} is the Hopf bundle over P_{m-1}. To see this, observe that $P_{k+m-1}-P_{k-1}=kH_{m-1}$, from which the statement follows immediately. Now $(k+2^{\phi(m-1)})H_{m-1}=kH_{m-1}\oplus 2^{\phi(m-1)}I$, where I is the trivial line bundle over P_{m-1} [1]. Hence $T((k+2^{\phi(m-1)})H_{m-1}) = \sum^{2^{\phi(m)}}T(kH_{m-1})$. Now the theorem follows from the suspension theorem.

REFERENCES

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2. I. M. James, *Cross sections of Stiefel manifolds*, Proc. London Math. Soc. (3) **8** (1958), 536–547.

3. J. H. C. Whitehead, *On the groups $\pi_r(V_{n,m})$ and sphere bundles*, Proc. London Math. Soc. (2) **48** (1944), 243–291.

NORTHWESTERN UNIVERSITY

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