THE RANGE OF Tf FOR CERTAIN LINEAR OPERATORS T

R. R. PHELPS

It is well known that any linear functional L on $C(X)$ (real or complex) such that $\|L\| = 1 = L(1)$ is *positive*, i.e., $Lf \geqq 0$ whenever $f \geqq 0$. More generally, this can be used to show that an operator T from $C(X)$ into $C(Y)$ is positive provided $\|T\| = 1$ and $T1 = 1$. The positivity of T is equivalent of course, to the assertion that if the range of f is contained in the nonnegative real axis, then so is the range of Tf. The theorem we prove below says more than this, in that it gives a description of the range of Tf in terms of the range of f, for arbitrary f. Furthermore, the proof is very simple and elementary.

Let A and B be linear spaces of bounded complex-valued functions on the sets X and Y, respectively, with the supremum norm. We assume that both A and B contain the constant functions. If Z is a subset of the complex plane, conv Z denotes the closed convex hull of Z.

THEOREM. *Suppose that T is a linear operator from A to B. Then $\|T\| = 1$ and $T1 = 1$ if and only if $(Tf)(Y) \subset \operatorname{conv} f(X)$ for each f in A. The operator T is an isometry and $T1 = 1$ if and only if* $\operatorname{conv}(Tf)(Y) = \operatorname{conv} f(X)$ *for each f in A.*

PROOF. The key to the proof is the observation that if K is a bounded closed convex set of complex numbers and $\beta \notin K$, then there exists a closed disc $\{z: |z-\alpha| \leqq r\}$ (α complex, $r > 0$) which contains K but not β. Thus, *if Z is a bounded subset of the plane, then* conv Z *is the intersection of all closed discs which contain Z.* Suppose, now, that $\|T\| = 1$, $T1 = 1$ and $f \in A$. If $f(X)$ is contained in the disc

$$\{z: |z - \alpha| \leqq r\},$$

then, for any y in Y, we have $r \geqq \|f - \alpha 1\| \geqq \|T(f - \alpha 1)\| = \|Tf - \alpha 1\| \geqq |(Tf)(y) - \alpha|$, so that $(Tf)(y)$ is in the same disc, and hence $(Tf)(Y) \subset \operatorname{conv} f(X)$. Conversely, if the latter is true for every f in A, then it is immediate that $T1 = 1$. Furthermore, for any f,

$$f(X) \subset \{z: |z| \leqq \|f\|\}.$$

Since $(Tf)(Y)$ is in this same disc, $\|Tf\| \leqq \|f\|$, and hence $\|T\| = 1$. Note that $(Tf)(Y) \subset \operatorname{conv} f(X)$ if and only if $\operatorname{conv}(Tf)(Y) \subset \operatorname{conv} f(X)$. If we interchange f and Tf in the above arguments, we see that (under the hypothesis that $T1 = 1$) the reverse inclusion is equivalent

Received by the editors December 19, 1963.

to $\|Tf\| \geqq \|f\|$. The assertion about isometries follows immediately from these facts.

COROLLARY. *If* $\|T\| = 1$ *and* $T1 = 1$, *then* $T \geqq 0$.

The above proof will yield the same theorem if the functions in A and B have their values in the same real normed linear space E, provided every bounded closed convex set in E is the intersection of all the closed balls which contain it. (The condition $T1 = 1$ becomes, of course, the condition that T is the identity map on the constant functions in A.) Spaces E with this property have been investigated in [1]; they include the Hilbert spaces (finite- and infinite-dimensional) as well as the spaces l^p and L^p, $1 < p < \infty$. In particular, the above theorem is true for spaces of real-valued functions.

REFERENCE

1. R. R. Phelps, *A representation theorem for bounded convex sets*, Proc. Amer. Math. Soc. 11 (1960), 976–983.

UNIVERSITY OF WASHINGTON